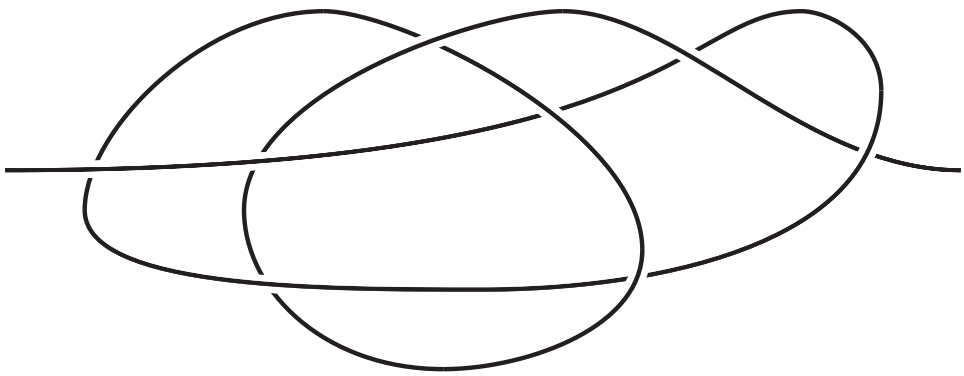
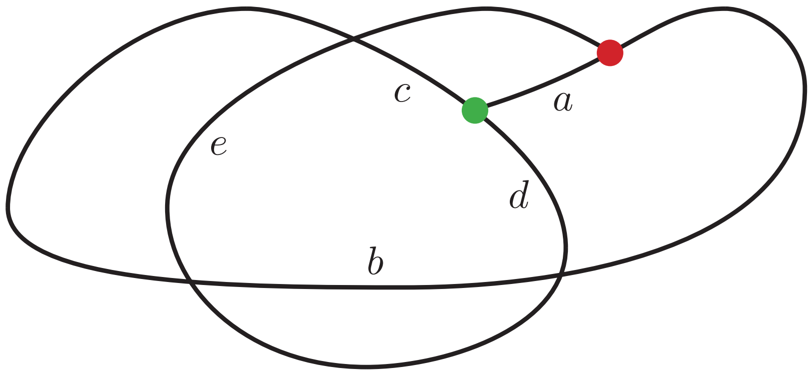
1. Introduction

Let D be a diagram of a hyperbolic knot K in S^3.



Remove an overpass and an underpass of D which are adjacent. Then, we have a subgraph G of D.



Put the *edge variables* and *dilogarithm functions* on the interior edges and corners of G respectively.

$$\mathrm{Li}_2(y/x) - \pi^2/6 \qquad \pi^2/6 - \mathrm{Li}_2(x/y)$$

Let $V(a,\dots,e)$ be the sum of these dilogarithms;

$$\begin{aligned}&\mathrm{Li}_2(a/d)-\mathrm{Li}_2(a/c)+\mathrm{Li}_2(a)-\mathrm{Li}_2(1/d)\\ &+\mathrm{Li}_2(b/d)-\mathrm{Li}_2(b)-\mathrm{Li}_2(1/b)+\mathrm{Li}_2(e/b)\\ &-\mathrm{Li}_2(e)-\mathrm{Li}_2(1/e)+\mathrm{Li}_2(c/e)-\mathrm{Li}_2(c)+\pi^2/3.\end{aligned}$$

Then, there is an ideal triangulation of $M=S^3\setminus K$, such that the hyperbolicity equations are given by

$$\begin{aligned}0&\equiv a\frac{\partial V}{\partial a}\equiv\ln\frac{1-a/c}{(1-a/d)(1-a)},\\ 0&\equiv b\frac{\partial V}{\partial b}\equiv\ln\frac{(1-b)(1-e/b)}{(1-b/d)(1-1/b)},\\ 0&\equiv c\frac{\partial V}{\partial c}\equiv\ln\frac{(1-a/c)(1-c)}{1-c/e},\\ 0&\equiv d\frac{\partial V}{\partial d}\equiv\ln\frac{(1-a/d)(1-b/d)}{1-1/d},\\ 0&\equiv e\frac{\partial V}{\partial e}\equiv\ln\frac{(1-e)(1-c/e)}{(1-e/b)(1-1/e)}\end{aligned}$$

modulo $2\pi\sqrt{-1}\mathbb{Z}$. Let $(a_0,\dots,e_0)$ denote the solution corresponding to the complete hyperbolic structure of M and define $\hat{V}(a,\dots,e)$ by

$$V(a,\dots,e)-\alpha\ln a-\beta\ln b-\gamma\ln c-\delta\ln d-\epsilon\ln e,$$

where

$$(\alpha,\dots,\epsilon)=\left(a\frac{\partial V}{\partial a},\dots,e\frac{\partial V}{\partial e}\right)_{(a,\dots,e)=(a_0,\dots,e_0)}.$$

Theorem.

$$-2\pi^2 \mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M) \equiv \hat{V}(a_0, \ldots, e_0) \mod \pi^2$$

- **Potential function**

```
li[x_] := PolyLog[2, x]
v[a_, b_, c_, d_, e_] := li[a / d] + li[a] + li[b / d] + li[e / b] + li[c / e] -
  li[a / c] - li[1 / d] - li[b] - li[1 / b] - li[e] - li[1 / e] - li[c] + π^2 / 3
```

- **Equations and solutions**

```
ev = {a, b, c, d, e};
dv = Table[Exp[ev[[i]] ∂ev[[i]] v[a, b, c, d, e]], {i, 5}] // Simplify
```

$$\left\{\frac{(-a+c)\,d}{(-1+a)\,c\,(a-d)},\ \frac{d\,(b-e)}{b-d},\ -\frac{(-1+c)\,c\,e}{(a-c)\,(c-e)},\ -\frac{(a-d)\,(-b+d)}{(-1+d)\,d},\ \frac{b\,(c-e)}{b-e}\right\}$$

```
sl = NSolve[Table[dv[[i]] == 1, {i, 5}]]
```

```
{{a → 1.06615 + 2.48427 ⅈ, e → -1.17738 + 0.250913 ⅈ, b → -1.09977 + 1.12945 ⅈ,
  c → -0.812447 - 0.173142 ⅈ, d → -0.0997713 + 1.12945 ⅈ},
 {a → 1.06615 - 2.48427 ⅈ, e → -1.17738 - 0.250913 ⅈ, b → -1.09977 - 1.12945 ⅈ,
  c → -0.812447 + 0.173142 ⅈ, d → -0.0997713 - 1.12945 ⅈ},
 {a → 1.28146 + 0.392024 ⅈ, e → 0.48782 - 0.110495 ⅈ,
  b → -0.317129 + 0.618084 ⅈ, c → 1.9499 + 0.441667 ⅈ,
  d → 0.682871 + 0.618084 ⅈ}, {a → 1.28146 - 0.392024 ⅈ,
  e → 0.48782 + 0.110495 ⅈ, b → -0.317129 - 0.618084 ⅈ,
  c → 1.9499 - 0.441667 ⅈ, d → 0.682871 - 0.618084 ⅈ},
 {a → 0.304778, e → 1.37912, b → 0.8338, c → 0.725102, d → 1.8338}}
```

```
Table[Im[ev[[i]] ∂ev[[i]] v[a, b, c, d, e]], {i, 5}] /. sl
```

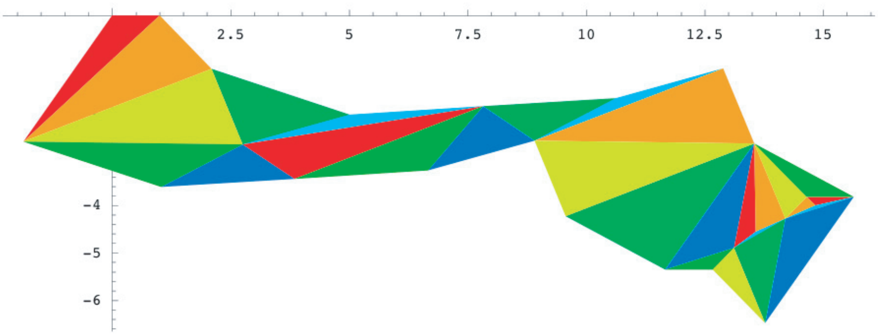
$\{\{-6.90093\times10^{-16}, -3.66199\times10^{-16}, 1.56746\times10^{-16},$
$1.34411\times10^{-15}, 3.31073\times10^{-16}\}, \{6.90093\times10^{-16}, 3.66199\times10^{-16},$
$-1.56746\times10^{-16}, -1.34411\times10^{-15}, -3.31073\times10^{-16}\},$
$\{-4.0423\times10^{-15}, -1.9194\times10^{-15}, 1.34793\times10^{-15}, 3.31151\times10^{-15},$
$-1.12598\times10^{-15}\}, \{4.0423\times10^{-15}, 1.9194\times10^{-15}, -1.34793\times10^{-15},$
$-3.31151\times10^{-15}, 1.12598\times10^{-15}\}, \{0, 0., 0, 0, 0.\}\}$

- **Complex volumes**

```
v[a, b, c, d, e] /. sl
```

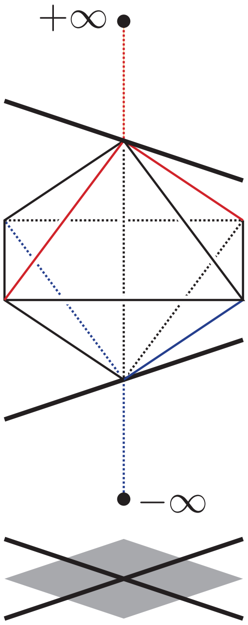
$\{11.9099 + 4.1249\,i, 11.9099 - 4.1249\,i, 1.85138 + 1.10891\,i,$
$1.85138 - 1.10891\,i, -1.20365 - 2.22045\times10^{-16}\,i\}$

- **Cusp torus**

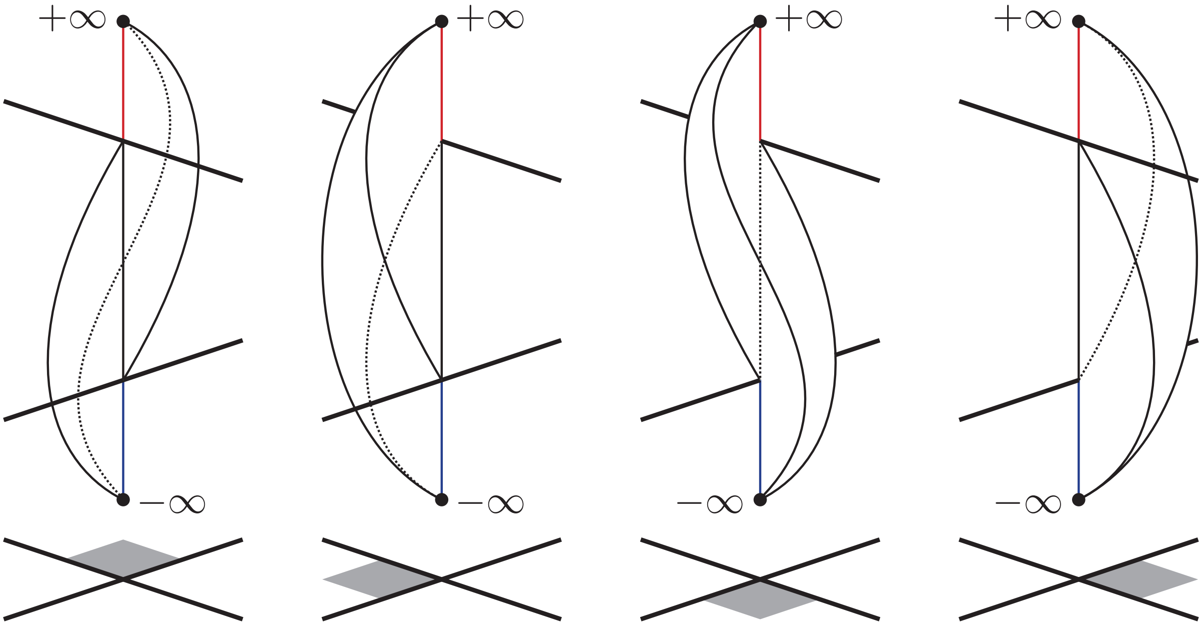


2. An ideal triangulation of M

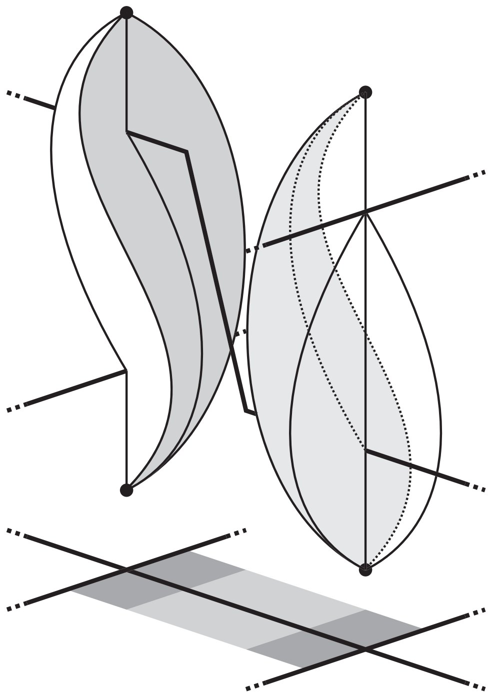
We prepare an ideal octahedron at each crossing of D, where $\pm\infty$ denote the poles of S^3.



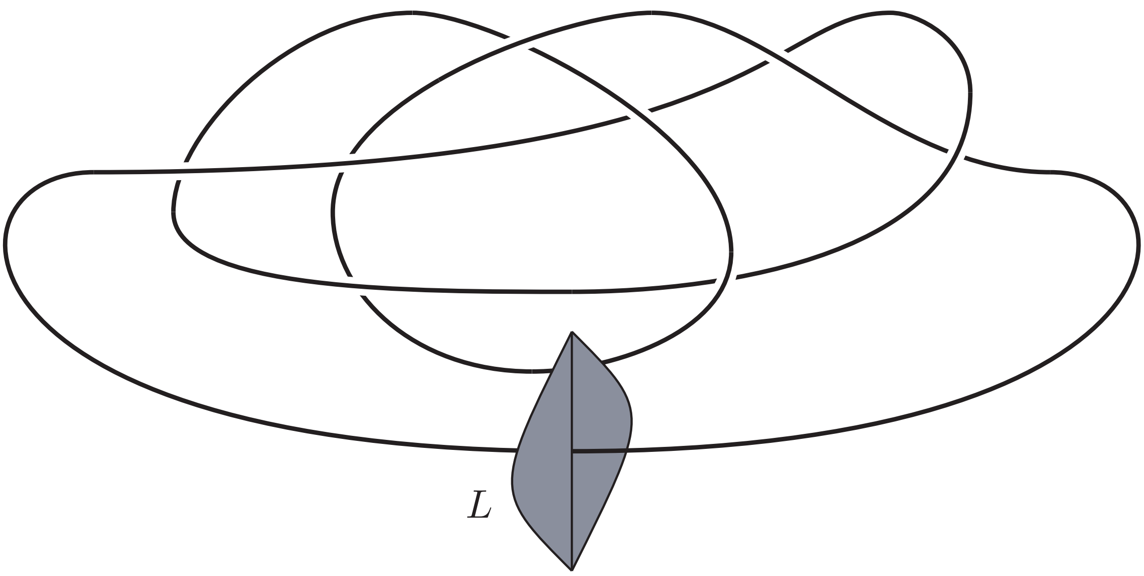
This octahedron decomposes into 4 tetrahedra.



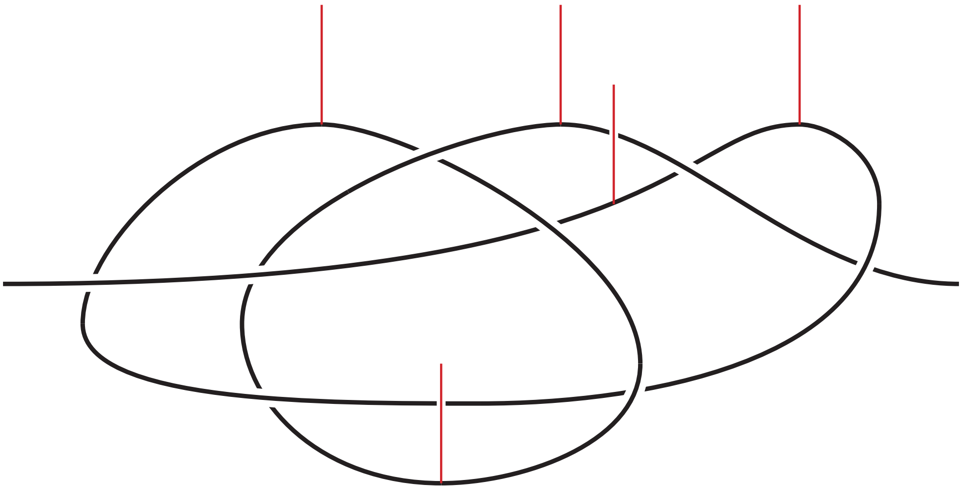
Glue them at the *leaves* corresponding to the edges of D. The result is $S^3 \setminus (K \cup \{\pm\infty\}) = M \setminus \{\pm\infty\}$.



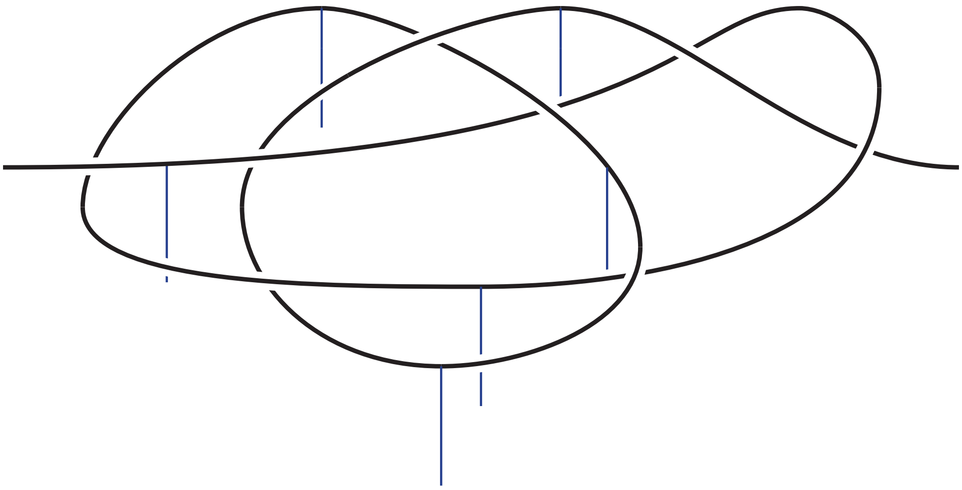
By contracting a leaf L below, we obtain an ideal triangulation $\mathcal{S}$ of M.



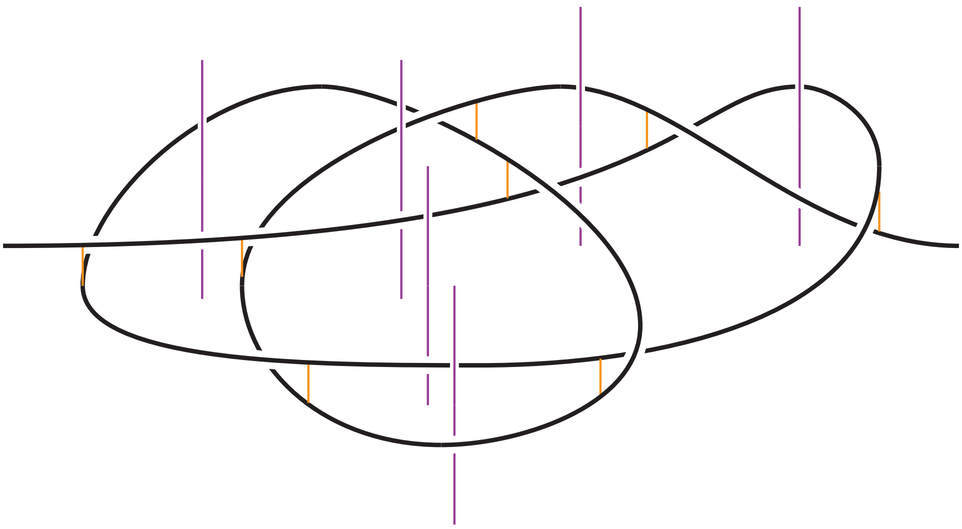
The edges of type A:



The edges of type B:

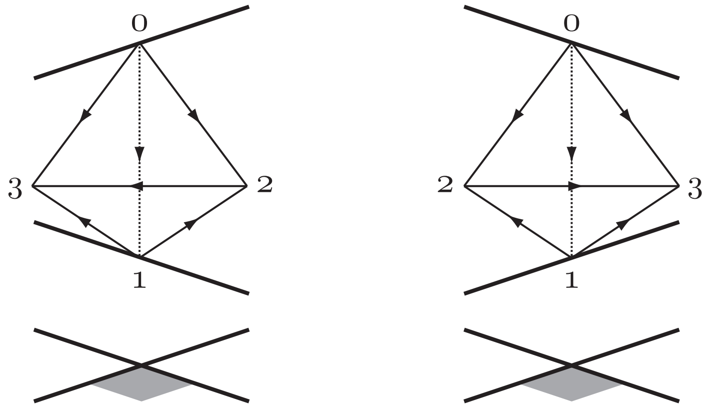


The others correspond to vertices and faces of G:



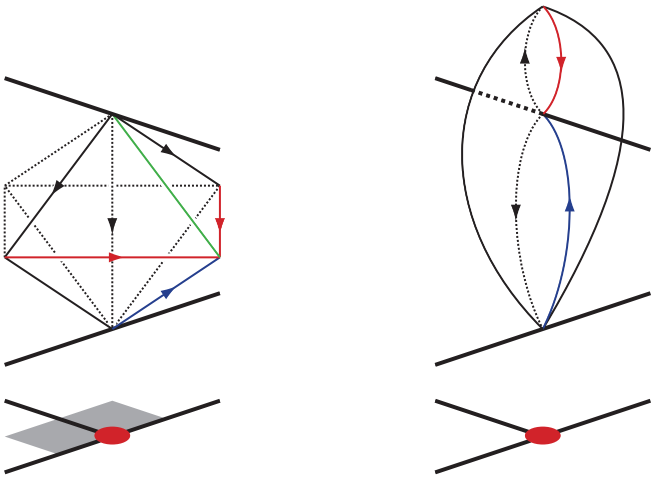
Vertex ordering

Order the vertices of each tetrahedron as follows.

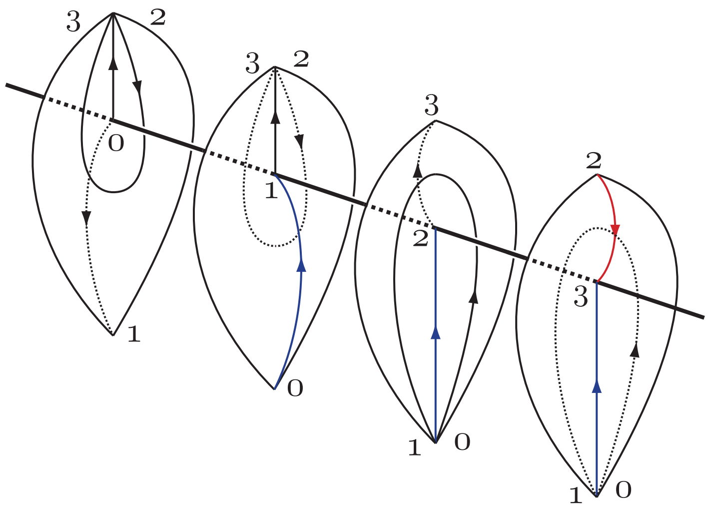


$\sigma(\tau) = 1$ $\sigma(\tau) = -1$

The orientation is consistent globally except:



So, we insert extra 4 ordered tetrahedra there.



3. Zickert's formula

Fix a cusp cross section of M. Then, for an edge x of $\mathcal{S}$, the holospheres around ∂x can be interchanged by an element of $\mathrm{PSL}_2(\mathbb{C})$ which is conjugate to

$$\begin{pmatrix} 0 & -1/\xi(x) \\ \xi(x) & 0 \end{pmatrix},$$

and we call $\xi(x)$ the *edge parameter* of x. Put

$$u(\tau) = \ln \xi(\tau_{03}) - \ln \xi(\tau_{31}) + \ln \xi(\tau_{12}) - \ln \xi(\tau_{20}),$$
$$v(\tau) = \ln \xi(\tau_{02}) - \ln \xi(\tau_{23}) + \ln \xi(\tau_{31}) - \ln \xi(\tau_{10})$$

for a tetrahedron τ of $\mathcal{S}$, where τ_{ij} denotes the edge of τ between the vertices i and j. If z denotes the modulus of τ at τ_{01}, there exist $p, q \in \mathbb{Z}$ such that

$$u(\tau) = \ln z^{\sigma(\tau)} + p\pi\sqrt{-1},$$
$$v(\tau) = \ln(1 - z^{\sigma(\tau)}) + q\pi\sqrt{-1}.$$

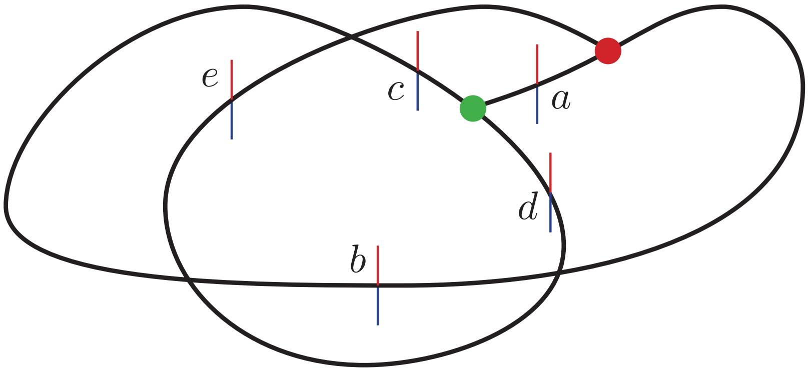
Then, we define $L(\tau)$ by

$$\mathrm{Li}_2(z^{\sigma(\tau)}) + \frac{1}{2}\ln z^{\sigma(\tau)} \ln(1 - z^{\sigma(\tau)}) - \frac{\pi^2}{6} + \frac{\pi\sqrt{-1}}{2}\left\{q \ln z^{\sigma(\tau)} + p\ln(1 - z^{\sigma(\tau)})\right\}.$$

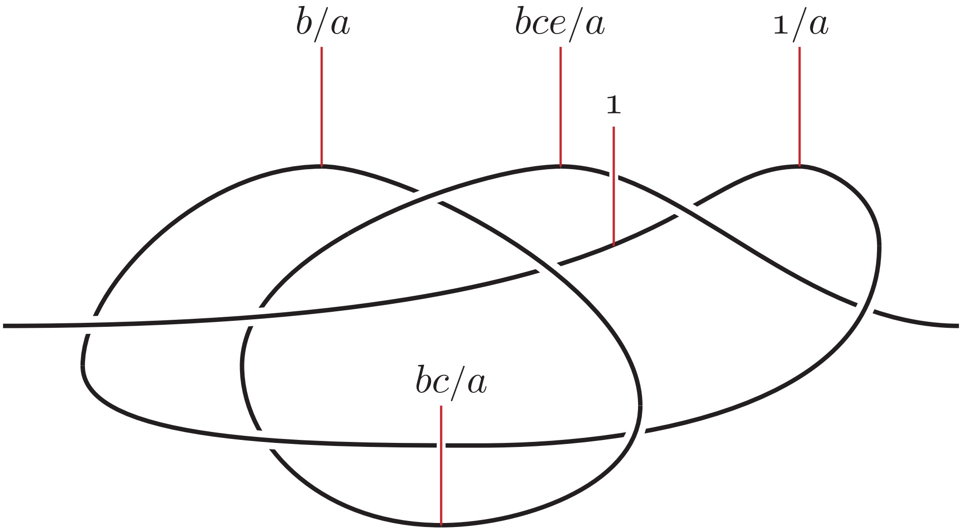
Zickert's Theorem.

$$\sum_{\tau} \sigma(\tau) L(\tau) \equiv -2\pi^2 \mathrm{cs}(M) + \sqrt{-1}\mathrm{vol}(M) \mod \pi^2$$

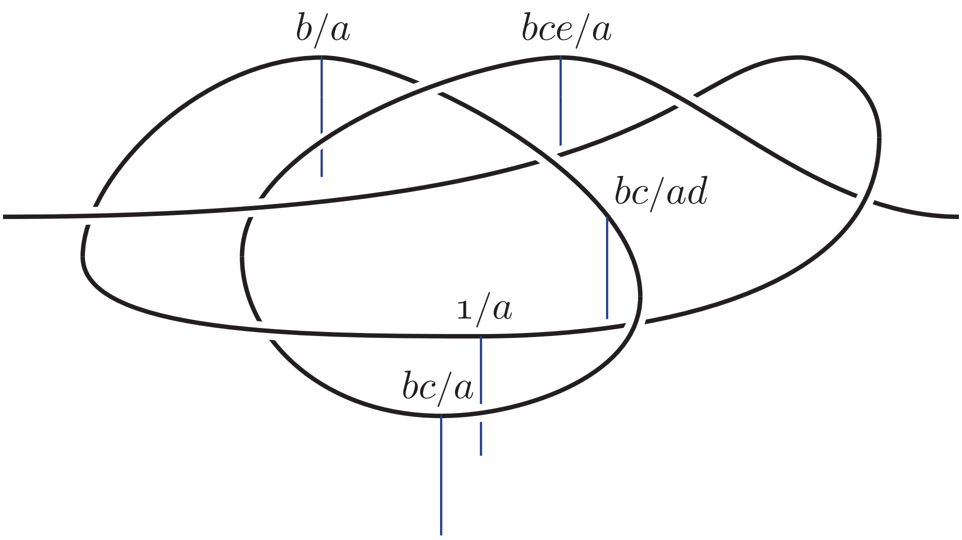
Lemma. *The ratios of the parameters of the edges of type A and B coincides with the edge variables.*



The parameters of the edges of type A:



The parameters of the edges of type B:



4. Proof

In our case, τ corresponds to a corner of G and z is the ratio of correponding edge variables. So, we can observe p is *even* by Lemma, and $L(\tau)$ is equal to

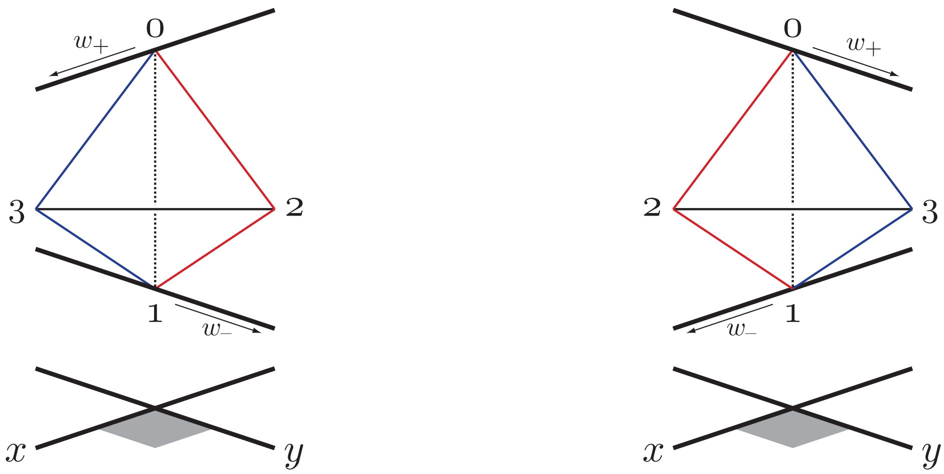
$$
\begin{aligned}
&\mathrm{Li}_2(z^{\sigma(\tau)}) - \frac{\pi^2}{6} \\
&+ \frac{1}{2}\left\{\ln z^{\sigma(\tau)} + p\pi\sqrt{-1}\right\}\left\{\ln(1 - z^{\sigma(\tau)}) + q\pi\sqrt{-1}\right\} \\
&= \mathrm{Li}_2(z^{\sigma(\tau)}) - \frac{\pi^2}{6} + \frac{1}{2}\,u(\tau)\left\{v(\tau) + 2\ln(1 - z^{\sigma(\tau)})\right\}
\end{aligned}
$$

modulo π^2. For simplicity, we put

$$w_+(\tau) = u_+(\tau)v_0(\tau), \quad w_-(\tau) = u_-(\tau)v_0(\tau),$$

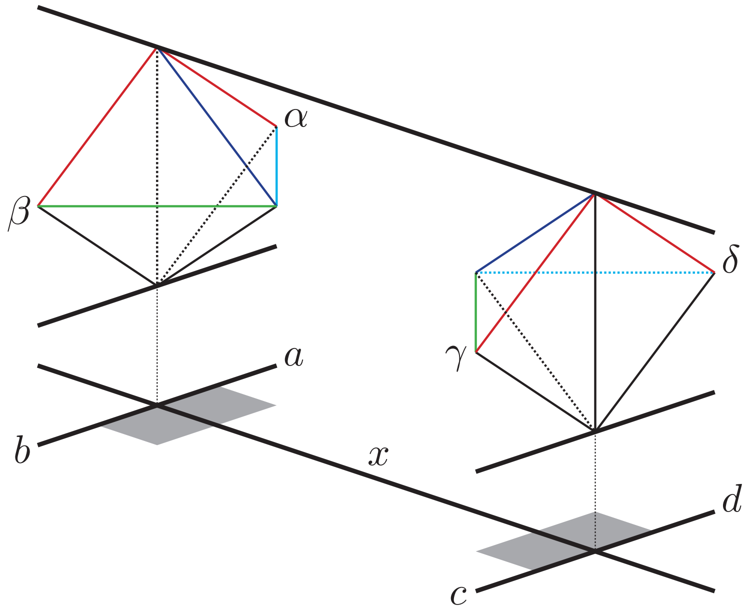
where $v_0(\tau) = v(\tau) + 2\ln(1 - z^{\sigma(\tau)})$ and

$$
\begin{aligned}
u_+(\tau) &= \ln\xi(\tau_{03}) - \ln\xi(\tau_{20}), \\
u_-(\tau) &= \ln\xi(\tau_{12}) - \ln\xi(\tau_{31}).
\end{aligned}
$$



$x = \frac{\xi(\tau_{20})}{\xi(\tau_{03})}$, $y = \frac{\xi(\tau_{12})}{\xi(\tau_{31})}$ $\qquad$ $x = \frac{\xi(\tau_{12})}{\xi(\tau_{31})}$, $y = \frac{\xi(\tau_{20})}{\xi(\tau_{03})}$

(1) Consider an edge e of G between *overcrossings*.



Then, $u_+(\alpha) = u_+(\beta) = u_+(\gamma) = u_+(\delta)$, and

$$\begin{aligned} w(e) &= w_+(\alpha) - w_+(\beta) + w_+(\gamma) - w_+(\delta) \\ &= u_+(\alpha)\{v_0(\alpha) - v_0(\beta) + v_0(\gamma) - v_0(\delta)\}, \end{aligned}$$

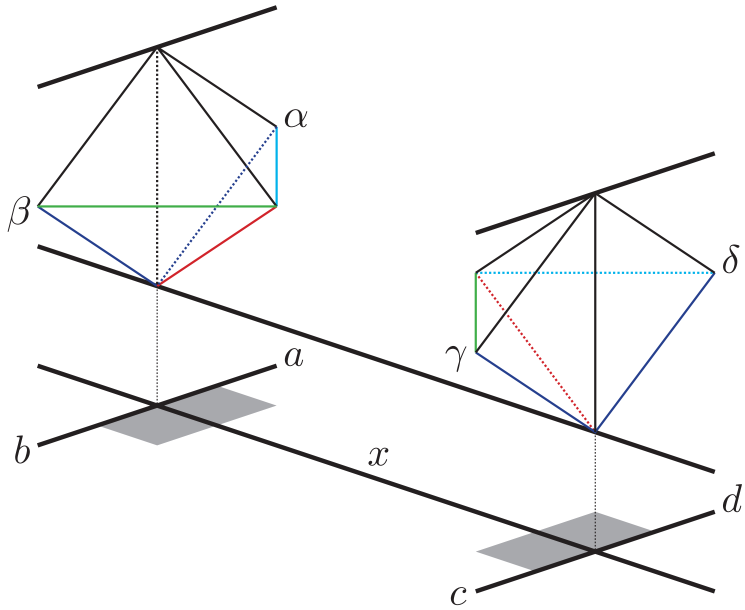
where $v_0(\alpha) - v_0(\beta) + v_0(\gamma) - v_0(\delta)$ is equal to

$$\begin{aligned} &\ln\xi(\alpha_{02}) - \ln\xi(\beta_{02}) + \ln\xi(\gamma_{02}) - \ln\xi(\delta_{02}) \\ &- \ln\xi(\alpha_{23}) + \ln\xi(\beta_{23}) - \ln\xi(\gamma_{23}) + \ln\xi(\delta_{23}) \\ &+ \ln\xi(\alpha_{31}) - \ln\xi(\beta_{31}) + \ln\xi(\gamma_{31}) - \ln\xi(\delta_{31}) \\ &- \ln\xi(\alpha_{10}) + \ln\xi(\beta_{10}) - \ln\xi(\gamma_{10}) + \ln\xi(\delta_{10}) \\ &+2\left\{\ln(1-\tfrac{a}{x}) - \ln(1-\tfrac{b}{x}) + \ln(1-\tfrac{c}{x}) - \ln(1-\tfrac{d}{x})\right\}. \end{aligned}$$

Therefore we have

$$\frac{1}{2}\,w(e) = u_+(\alpha)\cdot x\frac{\partial V}{\partial x} \equiv -\ln x\cdot x\frac{\partial V}{\partial x} \mod 4\pi^2.$$

(2) Consider an edge e of G between *undercrossings*.



Then, $u_-(\alpha) = u_-(\beta) = u_-(\gamma) = u_-(\delta)$, and

$$w(e) = -w_-(\alpha) + w_-(\beta) - w_-(\gamma) + w_-(\delta)$$
$$= -u_-(\alpha)\{v_0(\alpha) - v_0(\beta) + v_0(\gamma) - v_0(\delta)\},$$

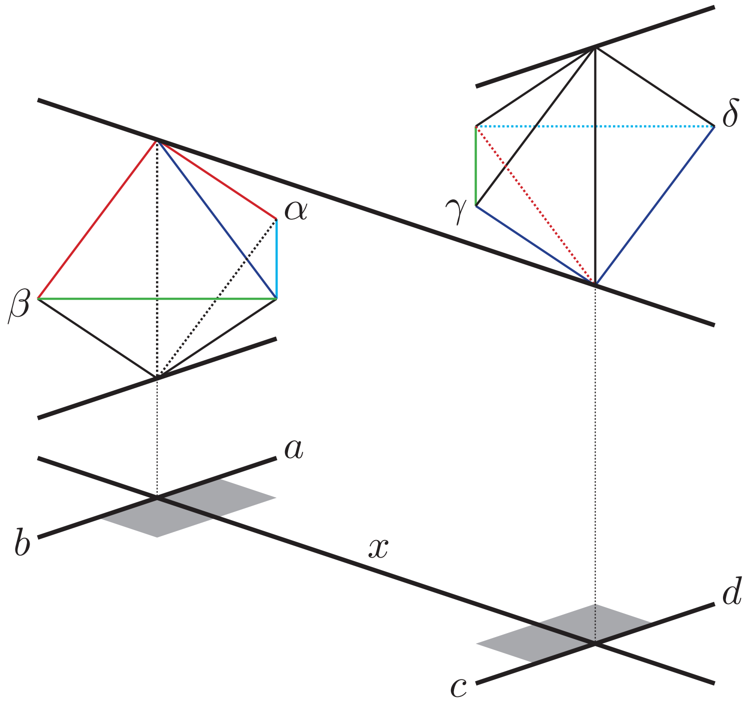
where $v_0(\alpha) - v_0(\beta) + v_0(\gamma) - v_0(\delta)$ is equal to

$$\begin{aligned} &\ln\xi(\alpha_{02}) - \ln\xi(\beta_{02}) + \ln\xi(\gamma_{02}) - \ln\xi(\delta_{02}) \\ &- \ln\xi(\alpha_{23}) + \ln\xi(\beta_{23}) - \ln\xi(\gamma_{23}) + \ln\xi(\delta_{23}) \\ &+ \ln\xi(\alpha_{31}) - \ln\xi(\beta_{31}) + \ln\xi(\gamma_{31}) - \ln\xi(\delta_{31}) \\ &- \ln\xi(\alpha_{10}) + \ln\xi(\beta_{10}) - \ln\xi(\gamma_{10}) + \ln\xi(\delta_{10}) \\ &+2\left\{\ln(1-\tfrac{x}{a}) - \ln(1-\tfrac{x}{b}) + \ln(1-\tfrac{x}{c}) - \ln(1-\tfrac{x}{d})\right\}. \end{aligned}$$

Therefore we have

$$\frac{1}{2}\,w(e) = -u_-(\alpha)\cdot x\frac{\partial V}{\partial x} \equiv -\ln x\cdot x\frac{\partial V}{\partial x} \mod 4\pi^2.$$

(3) Consider an edge e of G other than (1) and (2).



Then, $u_+(\alpha) = u_+(\beta) = -u_-(\gamma) = -u_-(\delta)$, and

$$\begin{aligned} w(e) &= w_+(\alpha) - w_+(\beta) - w_-(\gamma) + w_-(\delta) \\ &= u_+(\alpha)\{v_0(\alpha) - v_0(\beta) + v_0(\gamma) - v_0(\delta)\}, \end{aligned}$$

where $v_0(\alpha) - v_0(\beta) + v_0(\gamma) - v_0(\delta)$ is equal to

$$\begin{aligned} &\ln\xi(\alpha_{02}) - \ln\xi(\beta_{02}) + \ln\xi(\gamma_{02}) - \ln\xi(\delta_{02}) \\ &- \ln\xi(\alpha_{23}) + \ln\xi(\beta_{23}) - \ln\xi(\gamma_{23}) + \ln\xi(\delta_{23}) \\ &+ \ln\xi(\alpha_{31}) - \ln\xi(\beta_{31}) + \ln\xi(\gamma_{31}) - \ln\xi(\delta_{31}) \\ &- \ln\xi(\alpha_{10}) + \ln\xi(\beta_{10}) - \ln\xi(\gamma_{10}) + \ln\xi(\delta_{10}) \\ &+2\left\{\ln(1-\tfrac{a}{x}) - \ln(1-\tfrac{b}{x}) + \ln(1-\tfrac{x}{c}) - \ln(1-\tfrac{x}{d})\right\}. \end{aligned}$$

Therefore we have

$$\frac{1}{2}\,w(e) = u_+(\alpha)\cdot x\frac{\partial V}{\partial x} \equiv -\ln x\cdot x\frac{\partial V}{\partial x} \mod 4\pi^2.$$

5. Deformations

We can deform $V(a,\dots,e)$ so that a solution to

$$a\frac{\partial V}{\partial a},\ b\frac{\partial V}{\partial b},\ c\frac{\partial V}{\partial c},\ d\frac{\partial V}{\partial d},\ e\frac{\partial V}{\partial e} \in 2\pi\sqrt{-1}\mathbb{Z}$$

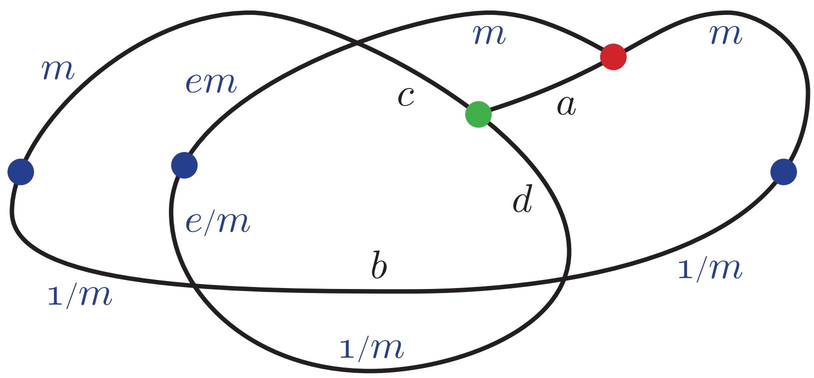
corresponds to a non-complete hyperbolic structure of M. In fact, such $V(a,\dots,e;m)$ is defined by

$$\begin{aligned}
&\mathrm{Li}_2(a/d) - \mathrm{Li}_2(a/c) + \mathrm{Li}_2(a/m) - \mathrm{Li}_2(1/dm)\\
&+\mathrm{Li}_2(b/d) - \mathrm{Li}_2(bm) - \mathrm{Li}_2(1/bm) + \mathrm{Li}_2(e/bm)\\
&-\mathrm{Li}_2(e) - \mathrm{Li}_2(1/e) + \mathrm{Li}_2(c/em) - \mathrm{Li}_2(c/m) + \pi^2/3\\
&+2\ln m\,(\ln b + \ln c + \ln e + 3\ln m),
\end{aligned}$$

where $m, 1/m$ are the eigenvalues of the holonomy of the *meridian* of K. Furthermore,

$$m\frac{\partial V}{\partial m} \equiv \ln\frac{(1-a/m)(1-bm)(1-b/em)(1-c/em)}{(1-1/dm)(1-1/bm)(1-c/m)} + 2\ln b + 2\ln c + 2\ln e + 12\ln m$$

equals $-2\ln\ell$ modulo $2\pi\sqrt{-1}\mathbb{Z}$, where $\ell, 1/\ell$ are the eigenvalues of the holonomy of the *longitude*.



Neumann-Zagier-Yoshida theory

Let $a(m), \ldots, e(m)$ be the geometric solution to

$$a\frac{\partial V}{\partial a},\ b\frac{\partial V}{\partial b},\ c\frac{\partial V}{\partial c},\ d\frac{\partial V}{\partial d},\ e\frac{\partial V}{\partial e} \in 2\pi\sqrt{-1}\mathbb{Z}.$$

Then,

$$\Phi(m) = V(a(m), \ldots, e(m); m) \\ - \alpha \ln a(m) - \cdots - \epsilon \ln e(m) - \mu \ln m$$

is the potential function of Neumann-Zagier, where

$$\mu = \left[m\frac{\partial V}{\partial m} \right]_{(a,\ldots,e,m)=(a_0,\ldots,e_0,1)}.$$

Furthermore, suppose there exists a pair $\kappa = (p, q)$ of coprime integers such that $m = m_\kappa$ satisfies

$$p \cdot 2\ln m_\kappa + q \cdot 2 \ln \ell(m_\kappa) = 2\pi\sqrt{-1}.$$

Topologically, the completion M_κ of M is the result of (p, q) *Dehn surgery* on K.

Yoshida's Theorem. *Define* $f(m)$ *by*

$$f(m) = \Phi(m) + \ln m \, \ln \ell(m).$$

Then, if $p^2 + q^2$ *is sufficiently large,*

$$\operatorname{Im} f(m_\kappa) = \operatorname{vol}(M_\kappa) + \frac{\pi}{2} \cdot \operatorname{length}(\gamma_\kappa),$$

$$\operatorname{Re} f(m_\kappa) = -2\pi^2 \operatorname{cs}(M_\kappa) - \frac{\pi}{2} \cdot \operatorname{torsion}(\gamma_\kappa),$$

where $\gamma_\kappa = M_\kappa - M$.

Complex volume of M_κ

Define $H_\kappa(a,\dots,e;m)$ by

$$V(a,\dots,e;m)-\alpha\ln a-\cdots-\epsilon\ln e-\mu\ln m$$
$$+\frac{\ln m\,(2\pi\sqrt{-1}-p\ln m)+s\pi^2}{q},$$

where s is an integer satisfying $ps\equiv 1 \mod q$. Then,

$$m\frac{\partial H_\kappa}{\partial m}=m\frac{\partial V}{\partial m}-\mu+\frac{2\pi\sqrt{-1}-p\cdot 2\ln m}{q}$$
$$=\frac{2\pi\sqrt{-1}-p\cdot 2\ln m-q\cdot 2\ln\ell(m)}{q}$$

and so $(a(m_\kappa),\dots,e(m_\kappa),m_\kappa)$ is a solution to

$$a\frac{\partial H_\kappa}{\partial a}=\cdots=e\frac{\partial H_\kappa}{\partial e}=m\frac{\partial H_\kappa}{\partial m}=0.$$

On the other hand, $\operatorname{Im}H_\kappa(a,\dots,e,m)$ is equal to

$$D(a/d)-D(a/c)+\cdots-D(c/m)$$
$$+\ln|a|\operatorname{Im}\frac{\partial H_\kappa}{\partial a}+\cdots+\ln|e|\operatorname{Im}\frac{\partial H_\kappa}{\partial e}+\ln|m|\operatorname{Im}\frac{\partial H_\kappa}{\partial m}$$

where $D(z)=\operatorname{Im}\operatorname{Li}_2(z)+\ln|z|\arg(1-z)$.

Theorem. *If p^2+q^2 is sufficiently large,*

$$H_\kappa(a(m_\kappa),\dots,e(m_\kappa);m_\kappa)$$

is equal to

$$-2\pi^2\mathrm{cs}(M_\kappa)+\sqrt{-1}\,\mathrm{vol}(M_\kappa)\mod \pi^2.$$

Proof. It suffices to show

$$f(m_\kappa) - H_\kappa(a(m_\kappa), \ldots, e(m_\kappa); m_\kappa)$$
$$= \frac{\pi\sqrt{-1}}{2}\left\{-2\pi^2 \mathrm{torsion}(\gamma_\kappa) + \sqrt{-1}\,\mathrm{length}(\gamma_\kappa)\right\}.$$

In fact, the left hand side is equal to

$$\ln m_\kappa \ln \ell(m_\kappa) - \frac{\ln m_\kappa\,(2\pi\sqrt{-1} - p\ln m_\kappa) + s\pi^2}{q}$$
$$= -\frac{\ln m_\kappa\,\{2\pi\sqrt{-1} - p\ln m_\kappa - q\ln \ell(m_\kappa)\} + s\pi^2}{q}$$
$$= -\frac{\pi\sqrt{-1}\ln m_\kappa + s\pi^2}{q}$$

and the right hand side is equal to

$$\frac{\pi\sqrt{-1}}{2}\cdot\{r\cdot 2\ln m_\kappa + s\cdot 2\ln \ell(m_\kappa)\}$$
$$= \frac{\pi\sqrt{-1}}{2}\cdot\frac{qr\cdot 2\ln m_\kappa + s(2\pi\sqrt{-1} - p\cdot 2\ln m_\kappa)}{q}$$
$$= \frac{\pi\sqrt{-1}}{2}\cdot\frac{2\pi s\sqrt{-1} - 2\ln m_\kappa}{q},$$

where r is an integer satisfying $ps - qr = 1$. $\square$