LECTURE NOTES CABLING

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Theorem 1. *Let $g = \gcd(r,s)$, $p = s/g$ and $\mathbf{N} = (N_0, \cdots, N_{g-1})$. The unnormalized colored Jones polynomial of the zero framed $(i;r,s)$-cabling of a banded link L with c components can be expressed as follows:*

$$\tilde{J}_{M_1,\dots,M_{i-1},\mathbf{N},\dots,M_c}(L^r_{i;s})(q) =$$
$$\prod_{j=1}^{g} q^{-pr\frac{N_j^2-1}{4}} \sum_{w=-\frac{|\mathbf{N}|-g}{2}}^{\frac{|\mathbf{N}|-g}{2}} \binom{g}{w}_{\mathbf{N}} q^{\frac{rw(wp+1)}{g}} J_{M_1,\dots,M_{i-1},2wp+1,\dots,M_c}(L)(q)$$

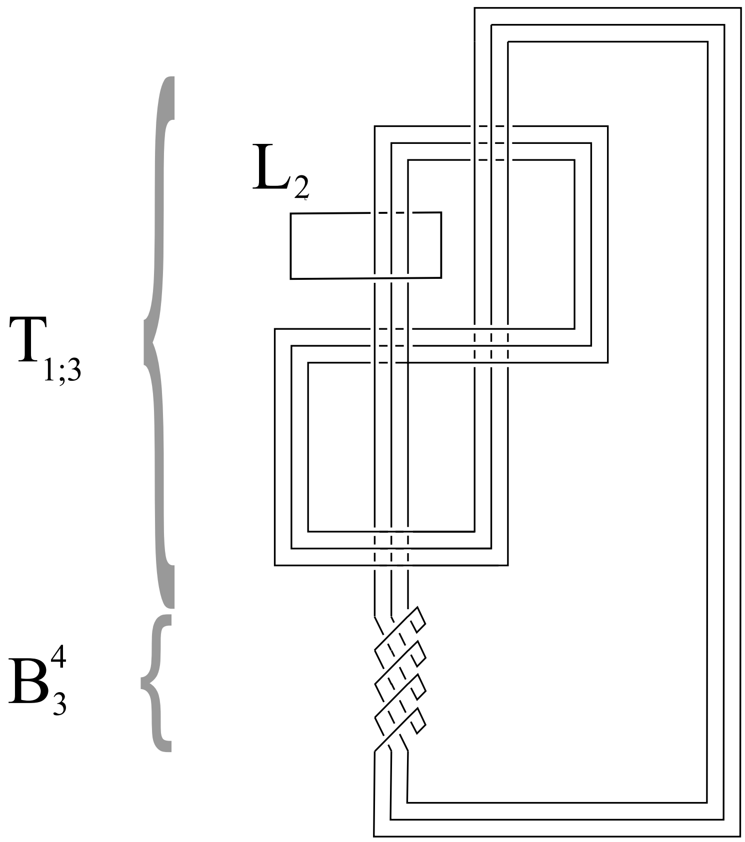

Figure 1. We have drawn the link $L^4_{1;3}$, the $(1;3,4)$-cabling of $L = (L_1, L_2)$, where L_1 is the figure eight knot and L_2 is an unknot. We have indicated the torus braid B^4_3 and the opened tangle $T_{1;3}$ mentioned in section 2.

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The above theorem works with a banded link or ribbon link L so that every component is an embedded annulus. Given a diagram D of a banded link inside an annulus we can construct a satellite of L by embedding D into a component L_i of L. The (r,s)-cabling operation is the special case where we take D to be the closure of the (r,s)-torus braid $B_s^r = (\sigma_1 \cdots \sigma_{s-1})^r$, where $r \in \mathbb{Z}, s \in \mathbb{N}$. To turn B_s^r into a banded tangle we use the blackboard framing and add a positive curl to every overpassing arc, see Figure 1 above. The banded link obtained by (r,s)-cabling the component L_i of a banded link L will be denoted by $L^r_{i;s}$, we will also call it the $(i;r,s)$-cabling of L.

In the above cabling formula we used to the following coefficients. For a vector $\mathbf{N} = (n_0, \ldots, n_{g-1})$ define $\binom{g}{w}_{\mathbf{N}}$ to be the coefficient of x^w in the expansion of the product

$$\prod_{k=0}^{g-1}(x^{\frac{N_k-1}{2}} + x^{\frac{N_k-1}{2}-1} + \ldots + x^{-\frac{N_k-1}{2}})$$

We also made use of the notation $|\mathbf{N}| = N_0 + \ldots + N_{g-1}$ and the convention that

$$J_{M_1,\ldots,M_{i-1},-j,\ldots,M_c}(L)(q) = -J_{M_1,\ldots,M_{i-1},j,\ldots,M_c}(L)(q)$$

Theorem 2. *Let $O^{\mathbf{r}}_{\mathbf{s}}$ be a zero framed iterated torus knot with cabling parameters $(\mathbf{r},\mathbf{s}) = (r_k, s_k)_{k=1}^K$.*

a. *Let $X = (X_k)_{k=1}^K$, $X_0 = 0$ and $r'_k = r_k - r_{k-1}s_{k-1}s_k$. Supposing $q = e^h$ and $h \notin \frac{2\pi i}{N}\mathbb{Z}$, we have*

$$\frac{\tilde{J}_N(O^{\mathbf{r}}_{\mathbf{s}})}{[N]}(q) = \frac{q^{-\frac{1}{4}(r_K s_K(N^2-1)+\frac{s_1}{r_1}+\sum_k \frac{r'_k}{s_k})}}{\sinh\frac{Nh}{2}\sqrt{(\pi h)^K \prod_k r'_k s_k}} \int_C e^{Q(X)} \sinh\frac{X_1}{r_1} \prod_k \frac{\sinh\frac{X_k}{s_k}}{\sinh X_k} dX$$

where Q is the following quadratic form:

$$Q(X) = \sum_k \frac{(X_k - s_k X_{k-1})^2}{-r'_k s_k h} + NX_K = \frac{1}{h}\tilde{Q}(X) + NX_K$$

and the contour C is a real K–dimensional plane such that $X_k - sX_{k-1} \in \mathbb{R}e^{\phi_k i}$ for some ϕ_k satisfying $\Re\frac{e^{2\phi i}}{-r'_k s_k h} < 0$

b. *Let $m \neq 0$ be an integer.*

$$\frac{\tilde{J}_N(O^{\mathbf{r}}_{\mathbf{s}})}{[N]}(e^{\frac{2m\pi i}{N}}) = N^{\frac{K}{2}+1}\frac{e^{-\frac{2m\pi i}{4N}(r_K s_K(N^2-1)+\frac{s_1}{r_1}+\sum_k \frac{r'_k}{s_k})}}{i^{\frac{K}{2}+2m}\pi^{K+2}m^{\frac{K}{2}+2}\sqrt{\prod_k r'_k s_k}}$$

$$\int_C e^{Q(X)}\tilde{Q}(X)\sinh\frac{X_1}{r_1}\prod_k \frac{\sinh\frac{X_k}{s_k}}{\sinh X_k}dX$$

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