LECTURE NOTES ASYMPTOTICS OF SPIN NETWORKS

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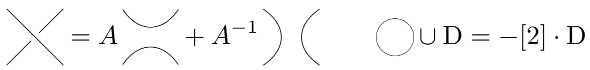
Definition 0.1. A *quantum spin network* (Γ, γ) consists of a *knotted ribbon graph* Γ (i.e., an embedded graph in 3-space, of arbitrary valency, together with a cyclic ordering of the edges around each vertex), together with a coloring $\gamma : \mathrm{Edges}(\Gamma) \longrightarrow \mathbb{N}$

Everything is expressed in terms of the Kauffman bracket. Set $A^4 = q$. Recall that *quantum integer* $[n]$ and the balanced *quantum factorial* $[n]!$ is defined by

$$[n] = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}, \qquad [n]! = \prod_{k=1}^{n} [k] \tag{1}$$



Definition 0.2. (a) We say a quantum spin network is *admissible* when the sum of the three labels a_v, b_v, c_v around every vertex v is even and a_v, b_v, c_v satisfy the triangle inequalities: $|a_v - b_v| \leq c_v \leq a_v + b_v$.
(b) The evaluation $\langle \Gamma, \gamma \rangle^P$ of a quantum spin network (Γ, γ) is defined to be zero if it is not admissible. An admissible quantum spin network is evaluated by the following algorithm.

- Use the cyclic ordering to thicken the vertices into disks and the edges into untwisted bands.
- Replace each vertex v by the pattern shown in Figure 1 and replace each edge by the linear combination of braids as shown.
- Finally the resulting linear combination of links is evaluated using the Kauffman bracket.

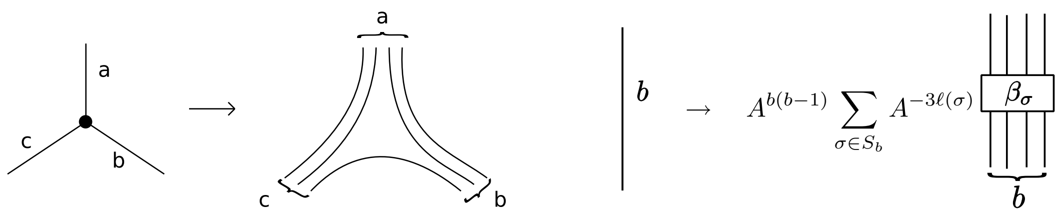


Figure 1. The rules for evaluating a quantum spin network. Replace vertices and edges to get a linear combination of links that is evaluated using the Kauffman bracket. Here b is the label of the edge, and for any permutation σ we denote by β_σ the unique negative permutation braid corresponding to σ. By $\ell(\sigma)$ we mean the minimal length of σ written as a product of transpositions.

(S) The *standard normalization* of a quantum spin network is defined by

$$\langle \Gamma, \gamma \rangle = \frac{1}{[\mathcal{I}]!} \langle \Gamma, \gamma \rangle^P$$

Here $\mathcal{I}! = \prod_{v \in V(\Gamma)} \left[\frac{-a_v+b_v+c_v}{2}\right]! \left[\frac{a_v-b_v+c_v}{2}\right]! \left[\frac{a_v+b_v-c_v}{2}\right]!$

(KL) We define the *Kauffman–Lins normalization* $\langle \Gamma, \gamma \rangle^{KL}$ to be

$$\langle \Gamma, \gamma \rangle^{KL} = \frac{1}{\prod_v [\gamma(v)]!} \langle \Gamma, \gamma \rangle^P$$

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(U) We define the *unitary normalization* $\langle\Gamma,\gamma\rangle^U$ to be

$$\langle\Gamma,\gamma\rangle^U=\frac{1}{\Theta(\gamma)}\langle\Gamma,\gamma\rangle \quad \text{where} \quad \Theta(\gamma)=\prod_{v\in V(\Gamma)}\sqrt{|\langle\Theta,a_v,b_v,c_v\rangle|}$$

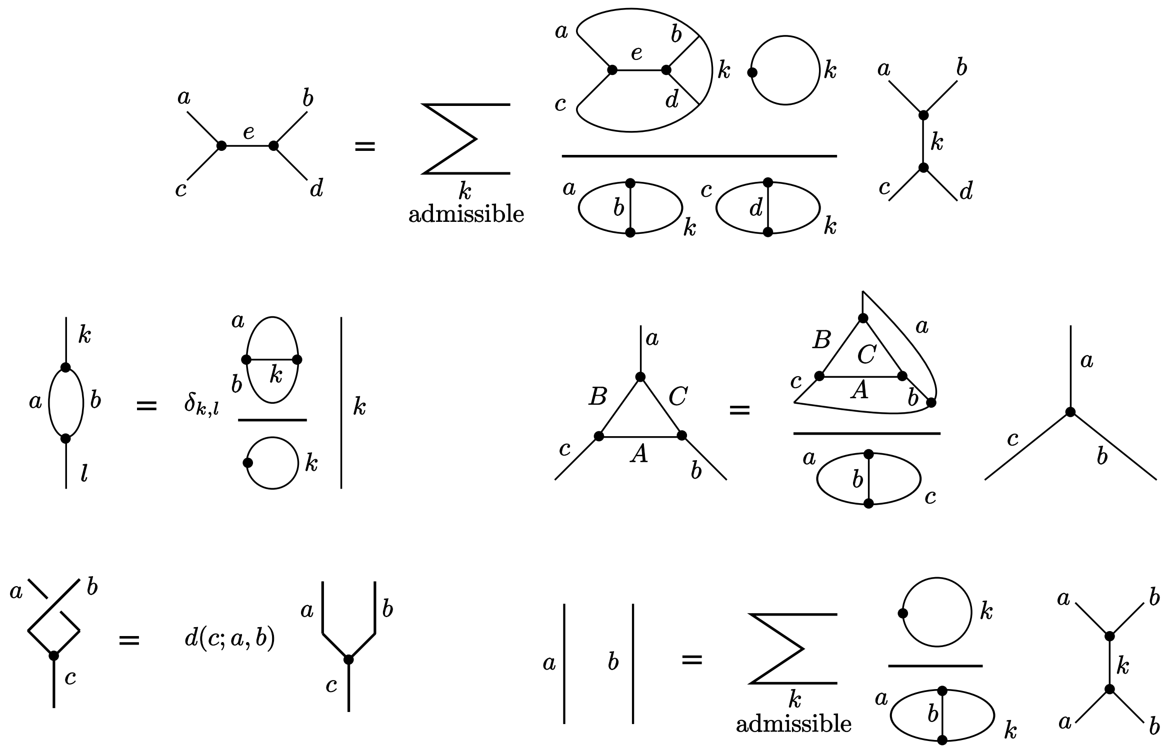

Figure 2. The formulas of recoupling theory. $d(c;a,b)=(-1)^{\frac{c-a-b}{2}}q^{\frac{c(c+2)-a(a+2)-b(b+2)}{8}}$ and $\delta_{k,l}$ is the *Kronecker delta function*.

Lemma 0.3. (a)

$$\langle\Theta,(a,b,c)\rangle=(-1)^{\frac{a+b+c}{2}}\left[\frac{a+b+c}{2}+1\right]\begin{bmatrix}\frac{a+b+c}{2}\\ \frac{-a+b+c}{2},\frac{a-b+c}{2},\frac{a+b-c}{2}\end{bmatrix} \tag{2}$$

(b) Let $(\triangle,\gamma)$ denote a tetrahedron labeled by $\gamma=(a,b,c,d,e,f)$ such that a is opposite to d, c to b and a,b,e makes a vertex. Then we have

$$\langle\triangle,\gamma\rangle=\sum_{k=\max T_i}^{\min S_j}(-1)^k[k+1]\begin{bmatrix}k\\ S_1-k,S_2-k,S_3-k,k-T_1,k-T_2,k-T_3,k-T_4\end{bmatrix} \tag{3}$$

where the S_i and T_j are given by

$$S_1=\frac{1}{2}(a+d+b+c) \qquad S_2=\frac{1}{2}(a+d+e+f) \qquad S_3=\frac{1}{2}(b+c+e+f) \tag{4}$$

$$T_1=\frac{1}{2}(a+b+e) \qquad T_2=\frac{1}{2}(a+c+f) \qquad T_3=\frac{1}{2}(c+d+e) \qquad T_4=\frac{1}{2}(b+d+f). \tag{5}$$

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