Quandle and hyperbolic volume

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1. Introduction

■ **quandle**

- an algebraic system
- homology theory

■ **hyperbolic volume (of 3-manifolds)**

- hyperbolic geometry
- cohomology class of $H^3(\mathrm{PSL}(2;\mathbb{C});\mathbb{R})$.

We will show that hyperbolic volume is a quandle 2-cocycle.

Further, we will show that the quandle 2-cocycle gives us a criterion for determining invertibility and amphicheirality of hyperbolic knots.

2. Preliminaries

Definition (quandle)

X : a non-empty set,

$* : X \times X \to X$ a binary operation.

$(X, *)$: a quandle

$\overset{\text{def}}{\Leftrightarrow}$ $*$ satisfies the following three axioms:

(Q1) ${}^{\forall}x \in X$, $x * x = x$.

(Q2) ${}^{\forall}y \in X$, $*\, y : X \to X \ \ (x \mapsto x * y)$ a bijection.

(Q3) ${}^{\forall}x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Example (conjugation quandle)

G : a group,
$X \subset G$: a subset closed under conjugations.
${}^{\forall}x, y \in X$, $x * y := y^{-1}xy$.

(Q1) ${}^{\forall}x \in X$, $x * x = x^{-1}xx = x$.

(Q2) ${}^{\forall}y, z \in X$, ${}^{\exists!}x \in X$ s.t. $x * y = z \quad (x = yzy^{-1})$.

(Q3) ${}^{\forall}x, y, z \in X$,

$$(x * y) * z = z^{-1}(y^{-1}xy)z = z^{-1}y^{-1}xyz,$$

$$(x * z) * (y * z) = (z^{-1}yz)^{-1}(z^{-1}xz)(z^{-1}yz) = z^{-1}y^{-1}xyz.$$

Definition (associated group)

X : a quandle.

$G_X := \langle x \in X \mid x * y = y^{-1}xy \ ({}^\forall x, y \in X) \rangle$

: the associated group of X.

More precisely,

$\mathcal{F}(X)$: a free group generated by the elements of X,

$\mathcal{N}(X)$: a subgroup of $\mathcal{F}(X)$ normally generated by

$$y^{-1}xy(x * y)^{-1} \quad ({}^\forall x, y \in X).$$

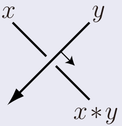
$G_X = F(X)/\mathcal{N}(X)$.

X : a quandle, Y : a set equipped with right action of G_X.

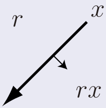
K : an oriented knot, D : a diagram of K.

Definition (shadow coloring)

- $\mathcal{A} : \{\text{arcs of } D\} \to X$ an arc coloring

  $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{A}$ satisfies the condition  at each crossing.

- $\mathcal{R} : \{\text{regions of } D\} \to Y$ a region coloring

  $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{R}$ satisfies the condition  around each arc.

- $\mathcal{S} := (\mathcal{A}, \mathcal{R})$: a shadow coloring.

A finite sequence of Reidemeister moves transforms a shadow coloring $\mathcal{S}$ into a unique shadow coloring $\mathcal{S}'$.

Therefore, the multi-set

$$\{\mathcal{S} : \text{a shadow coloring of } D \text{ w.r.t. } X \text{ and } Y\ \}$$

does not depend on the choice of a diagram D of K.

Using quandle 2-cocycle, we can refinement this set.

Quandle homology

X : a quandle.

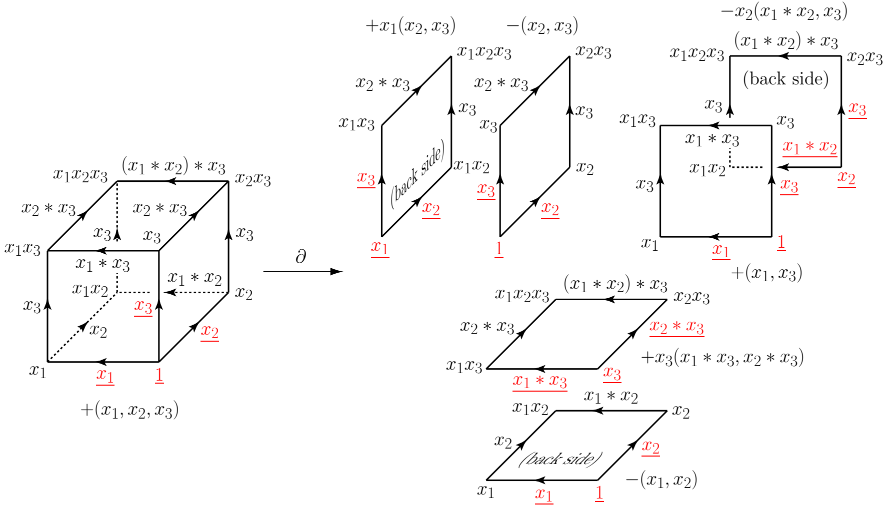
$C_n^R(X) := \mathrm{span}_{\mathbb{Z}[G_X]}\{(x_1, x_2, \cdots, x_n) \in X^n\}$.

Define $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial(x_1, \cdots, x_n) = \sum_{i=1}^{n} (-1)^i \{(x_1, \cdots, \widehat{x_i}, \cdots, x_n) - x_i(x_1 * x_i, \cdots, x_{i-1} * x_i, x_{i+1}, \cdots, x_n)\}.$$

$\rightsquigarrow (C_n^R(X), \partial)$: a chain complex.

$$\partial(x_1, \cdots, x_n) = \sum_{i=1}^{n} (-1)^i \{ (x_1, \cdots, \widehat{x_i}, \cdots, x_n) - x_i (x_1 * x_i, \cdots, x_{i-1} * x_i, x_{i+1}, \cdots, x_n) \}.$$



$$C_n^R(X) = \mathrm{span}_{\mathbb{Z}[G_X]}\{(x_1, x_2, \cdots, x_n) \in X^n\}.$$
$$\cup$$
$$C_n^D(X) := \mathrm{span}_{\mathbb{Z}[G_X]}\{(x_1, x_2, \cdots, x_n) \in X^n \mid {}^{\exists}i \text{ s.t. } x_i = x_{i+1}\}.$$

$$C_n^Q(X) := C_n^R(X)/C_n^D(X).$$

Definition (quandle homology group)

M : a right $\mathbb{Z}[G_X]$-module,

$C_n^Q(X; M) := M \otimes_{\mathbb{Z}[G_X]} C_n^Q(X)$.

$H_n^Q(X; M) := H_n(C_*^Q(X; M))$: the quandle homology group.

Definition (quandle cohomology group)

N : a left $\mathbb{Z}[G_X]$-module,

$C_Q^n(X; N) := \mathrm{Hom}_{\mathbb{Z}[G_X]}(C_n^Q(X), N)$.

$H_Q^n(X; N) := H^n(C_Q^*(X; N))$: the quandle cohomology group.

Remark

X : a quandle,
Y : a set equipped with a right action of G_X,
A : an abelian group.

$\rightsquigarrow \mathbb{Z}[Y]$: a right $\mathbb{Z}[G_X]$-module,
$\mathrm{Hom}(\mathbb{Z}[Y], A)$: a left $\mathbb{Z}[G_X]$-module.

$\rightsquigarrow H_*^Q(X; \mathbb{Z}[Y])$, $H_Q^*(X; \mathrm{Hom}(\mathbb{Z}[Y], A))$.

Define $\langle\ ,\ \rangle : H_Q^n(X; \mathrm{Hom}(\mathbb{Z}[Y], A)) \otimes H_n^Q(X; \mathbb{Z}[Y]) \to A$ by

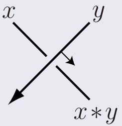
$$\langle f, r \otimes (x_1, x_2, \cdots, x_n)\rangle := f(x_1, x_2, \cdots, x_n)(r).$$

X : a quandle, Y : a set equipped with right action of G_X.

K : an oriented knot, D : a diagram of K.

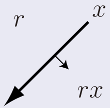
Definition (shadow coloring)

- $\mathcal{A} : \{\text{arcs of } D\} \to X$ an arc coloring

  $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{A}$ satisfies the condition 

  at each crossing.

- $\mathcal{R} : \{\text{regions of } D\} \to Y$ a region coloring

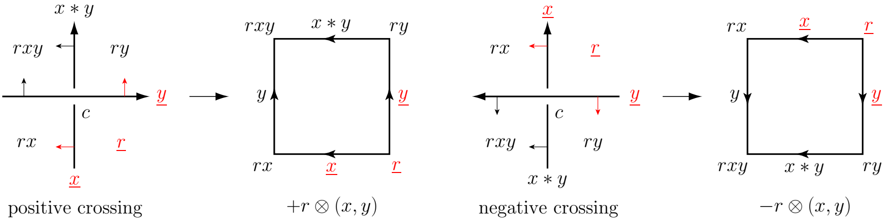
  $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{R}$ satisfies the condition 

  around each arc.

- $\mathcal{S} := (\mathcal{A}, \mathcal{R})$: a shadow coloring.

$\mathcal{S}$: a shadow coloring of D w.r.t. X and Y.

$$C(\mathcal{S}) := \sum_{\text{crossings}} \pm r \otimes (x, y) \quad \in C_2^Q(X; \mathbb{Z}[Y]).$$



Proposition (Carter-Kamada-Saito '01)

$C(\mathcal{S}) \in C_2^Q(X; \mathbb{Z}[Y])$ is a cycle.

Definition (fundamental class)

$[C(\mathcal{S})] \in H_2^Q(X; \mathbb{Z}[Y])$: the fundamental class of $\mathcal{S}$.

Theorem (Carter-Kamada-Saito '01)

$\mathcal{S}$, $\mathcal{S}'$: *shadow colorings related with each other by Reidemeister moves.*

$\Rightarrow [C(\mathcal{S})] = [C(\mathcal{S}')]$.

A : an abelian group.

$\theta \in C^2_Q(X; \mathrm{Hom}(\mathbb{Z}[Y], A))$: a cocycle, fix.

Then the multi-set

$$\{\langle [\theta], [C(\mathcal{S})] \rangle \in A \mid \mathcal{S} : \text{a shadow coloring of } D \text{ w.r.t. } X \text{ and } Y \}$$

does not depend on the choice of a diagram D of K.

We call the multi-set a quandle cocycle invariant of K.

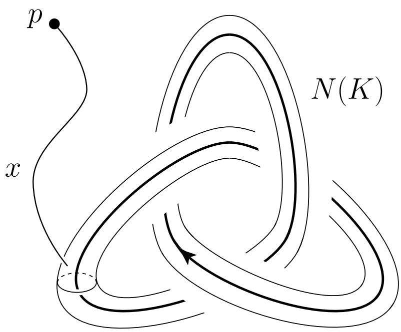
3. Hyperbolic volume is a quandle 2-cocycle

K : an oriented knot in S^3,

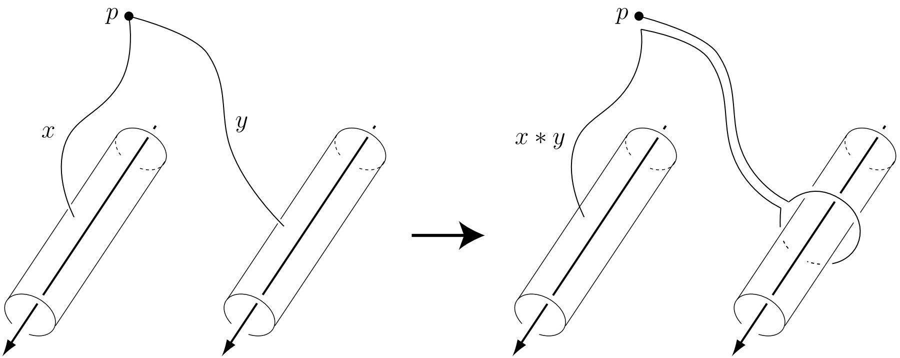
$N(K)$: a regular n.b.h.d. of K.

$p \in S^3 \setminus K$ fix.

$Q(K) := \{x : \text{a path from } p \text{ to } N(K)\}/\text{homotopy}.$



$Q(K)$ is a quandle with $[x] * [y] := [x * y] \quad ({}^{\forall}[x], [y] \in Q(K))$.

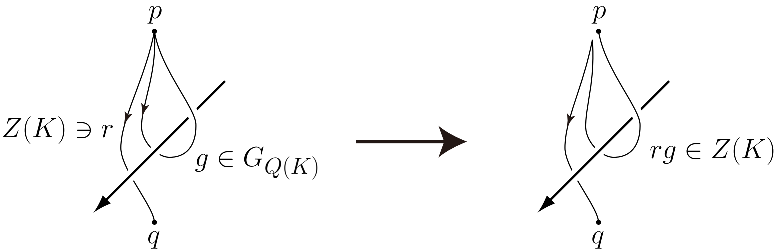


We call $Q(K)$ the knot quandle of K.

$q \in S^3 \setminus K$ fix.

$Z(K) := \{r : \text{a path from } p \text{ to } q\}/\text{homotopy}.$

$G_{Q(K)}(= \pi_1(S^3 \setminus K))$ acts on $Z(K)$ from the right.

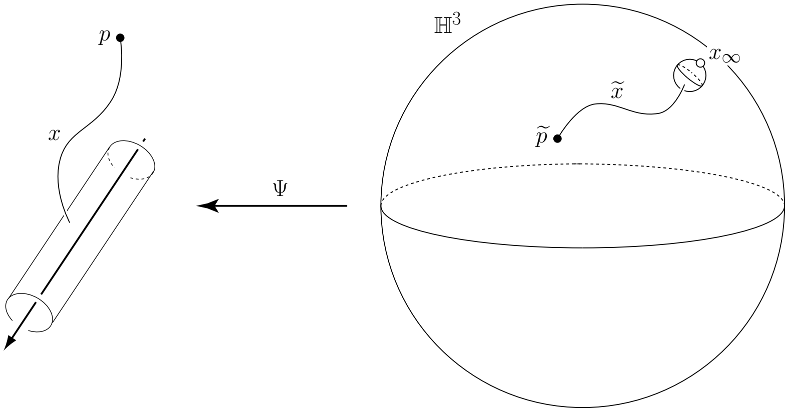


K : an oriented hyperbolic knot in S^3,

$\Psi : \mathbb{H}^3 \to S^3 \setminus K$ the universal covering.

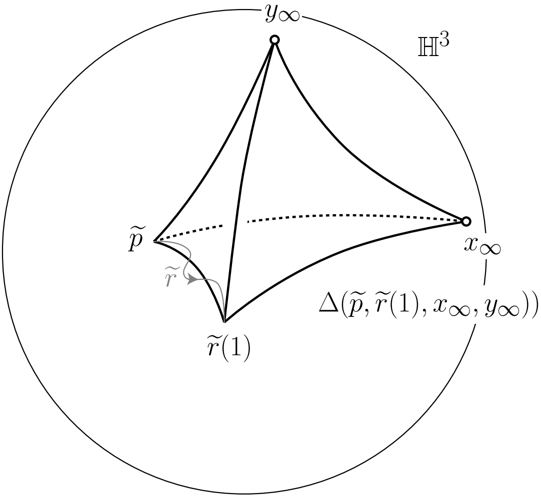
$\widetilde{p} \in \Psi^{-1}(p)$ fix.

We have a map $Q(K) \to \partial\overline{\mathbb{H}^3}$ which $x \mapsto x_\infty$.



$^\forall r \in Z(K)$, $^\forall x, y \in Q(K)$,

we can construct an oriented tetrahedron $\Delta(\widetilde{p}, \widetilde{r}(1), x_\infty, y_\infty)$.



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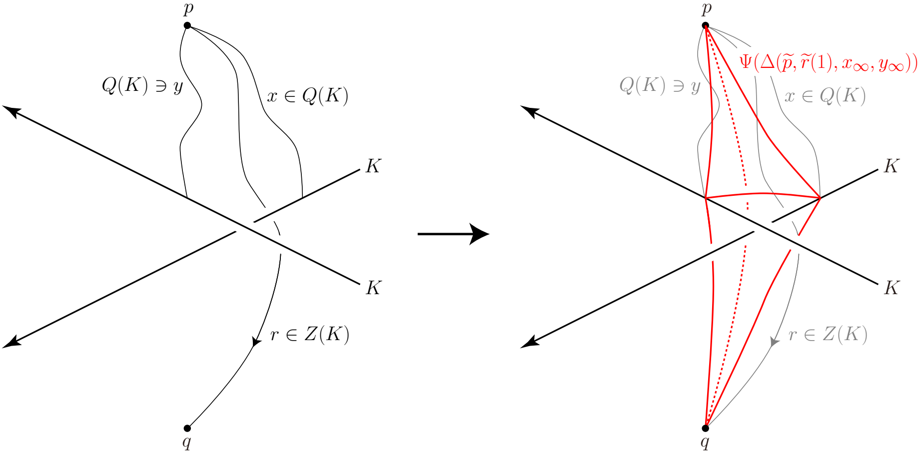
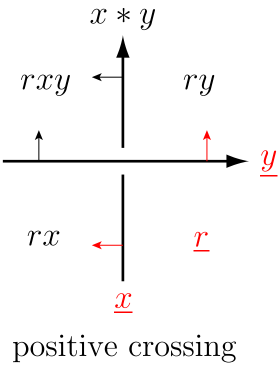
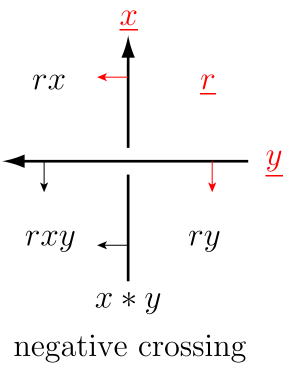
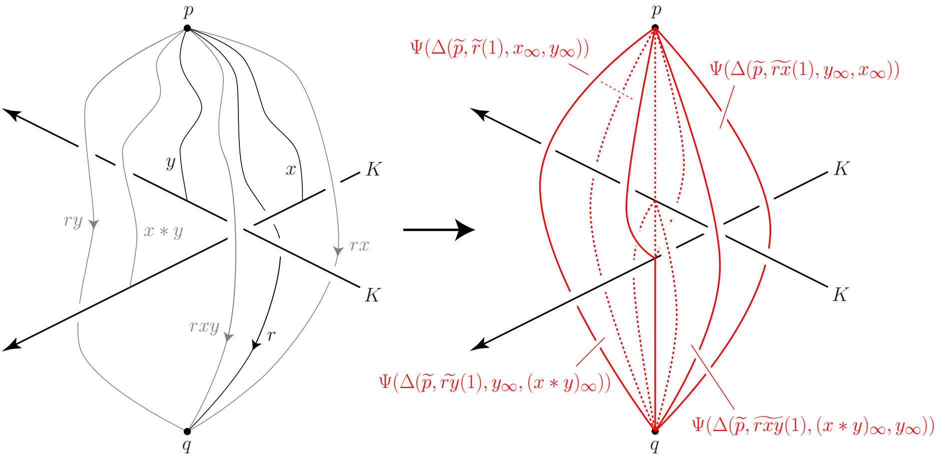


${}^{\forall}r \in Z(K)$, ${}^{\forall}x, y \in Q(K)$,

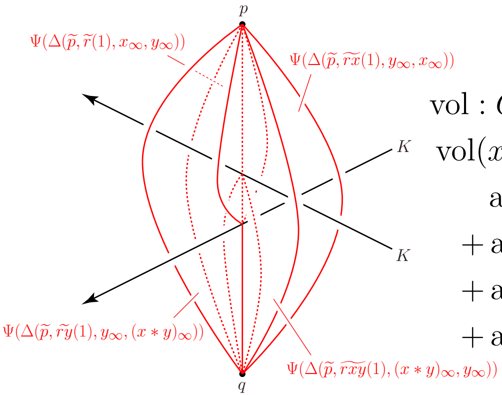




$^\forall r \in Z(K)$, $^\forall x, y \in Q(K)$,



$^{\forall}r \in Z(K)$, $^{\forall}x, y \in Q(K)$,



$$\mathrm{vol} : Q(K) \times Q(K) \to \mathrm{Hom}(\mathbb{Z}[Y], \mathbb{R})$$

$$\begin{aligned} \mathrm{vol}(x, y)(r) := &\ \mathrm{algvol}(\Delta(\widetilde{p}, \widetilde{r}(1), x_\infty, y_\infty)) \\ &+ \mathrm{algvol}(\Delta(\widetilde{p}, \widetilde{rx}(1), y_\infty, x_\infty)) \\ &+ \mathrm{algvol}(\Delta(\widetilde{p}, \widetilde{rxy}(1), (x*y)_\infty, y_\infty)) \\ &+ \mathrm{algvol}(\Delta(\widetilde{p}, \widetilde{ry}(1), y_\infty, (x*y)_\infty)). \end{aligned}$$

Theorem

$\mathrm{vol} \in C^2_Q(Q(K); \mathrm{Hom}(\mathbb{Z}[Z(K)], \mathbb{R}))$ *is a cocycle.*

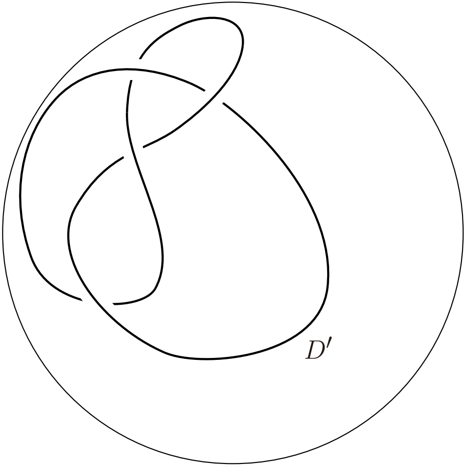
Theorem

K : *an oriented hyperbolic knot in* S^3.
$\mathrm{vol} \in C^2_Q(Q(K); \mathrm{Hom}(\mathbb{Z}[Z(K)], \mathbb{R}))$ *the above cocycle.*

K' : *an oriented knot in* S^3,
D' : *a diagram of* K'.

$^\forall \mathcal{S}$: *a shadow coloring of* D' *w.r.t.* $Q(K)$ *and* $Z(K)$,
$^\exists f_{\mathcal{S}} : S^3 \setminus K' \to S^3 \setminus K$ *a continuous map s.t.*
$\langle [\mathrm{vol}], [C(\mathcal{S})] \rangle = \deg(f_{\mathcal{S}}) \cdot \mathrm{vol}(S^3 \setminus K)$.

Outline of the proof



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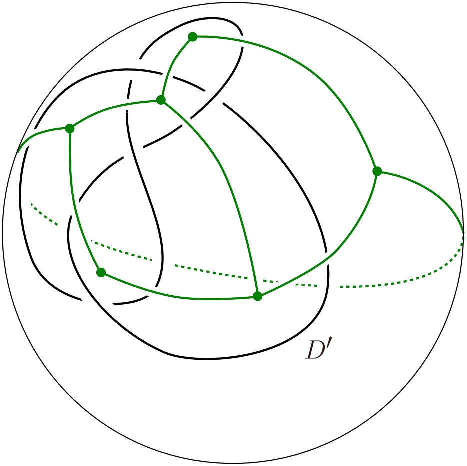

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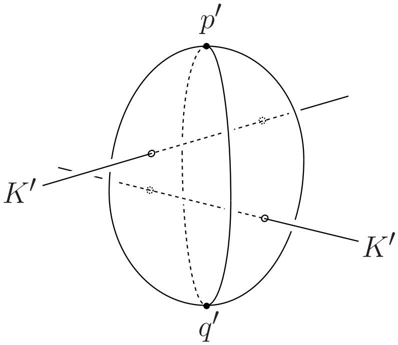

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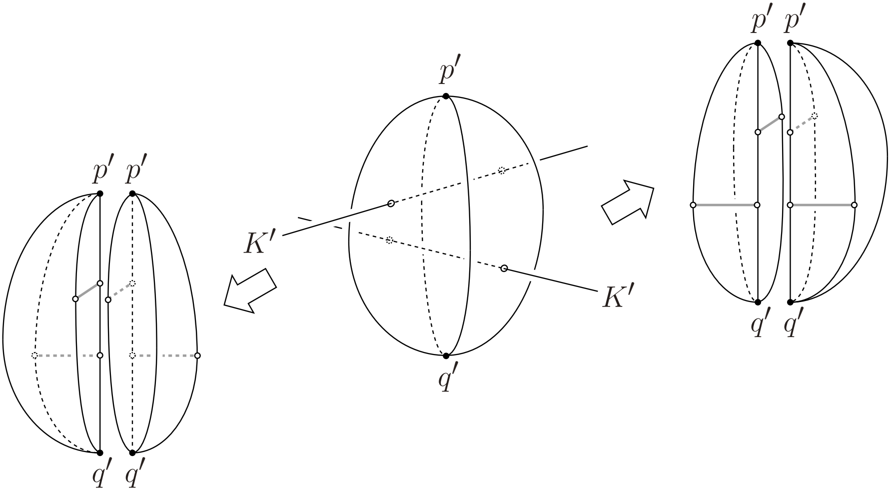

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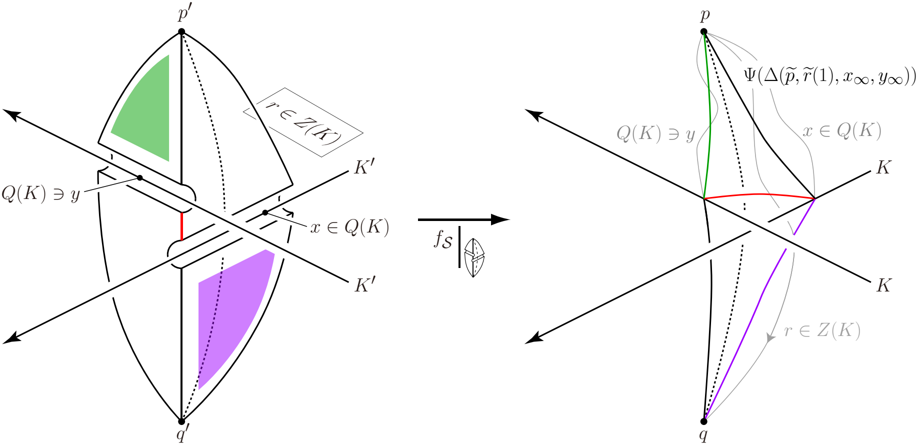

□

4. Invertibility and amphicheirality of hyperbolic knots

K : an oriented hyperbolic knot in S^3.

$\mathrm{vol} \in Z^2_Q(Q(K); \mathrm{Hom}(\mathbb{Z}[Z(K)], \mathbb{R}))$ as above.

Proposition

$K' = K \text{ or } -K$,

D' : a diagram of K'.

${}^\forall\mathcal{S}$: a shadow coloring of D' w.r.t. $Q(K)$ and $Z(K)$,

$\langle [\mathrm{vol}], [C(\mathcal{S})] \rangle = \pm \mathrm{vol}(S^3 \setminus K) \text{ or } 0$.

Proposition

D : a diagram of K.

${}^\exists\mathcal{S}$: a shadow coloring of D w.r.t. $Q(K)$ and $Z(K)$ s.t.

$\langle [\mathrm{vol}], [C(\mathcal{S})] \rangle = \mathrm{vol}(S^3 \setminus K)$.

Theorem

K : *an oriented hyperbolic knot in* S^3,

D : *a diagram of* K, $-D$: *a diagram of* $-K$.

(i) ${}^{\exists}\mathcal{S}$: *a shadow coloring of* D *w.r.t.* $Q(K)$ *and* $Z(K)$ *s.t.*
$\langle [\mathrm{vol}], [C(\mathcal{S})] \rangle = -\mathrm{vol}(S^3 \setminus K)$
$\Leftrightarrow K$ *is negative amphicheiral* (*i.e.* $K \cong -K^*$).

(ii) ${}^{\exists}\mathcal{S}$: *a shadow coloring of* $-D$ *w.r.t.* $Q(K)$ *and* $Z(K)$ *s.t.*
$\langle [\mathrm{vol}], [C(\mathcal{S})] \rangle = \mathrm{vol}(S^3 \setminus K)$
$\Leftrightarrow K$ *is invertible* (*i.e.* $K \cong -K$).

(iii) ${}^{\exists}\mathcal{S}$: *a shadow coloring of* $-D$ *w.r.t.* $Q(K)$ *and* $Z(K)$ *s.t.*
$\langle [\mathrm{vol}], [C(\mathcal{S})] \rangle = -\mathrm{vol}(S^3 \setminus K)$
$\Leftrightarrow K$ *is positive amphicheiral* (*i.e.* $K \cong K^*$).