Volume conjecture of colored Jones polynomials

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January 15, 2010

Introduction

Conjecture (Volume conjecture of Kashaev invariant)

$$\mathrm{vol}(L) = 2\pi \lim_{N\to\infty} \frac{\log|\langle L\rangle_N|}{N},$$

where L is a hyperbolic link, $\mathrm{vol}(L)$ is the hyperbolic volume, $\langle L\rangle_N$ is the Kashaev invariant.

Conjecture (Complexified volume conjecture)

$$i(\mathrm{vol}(L) + i\,\mathrm{cs}(L)) \equiv 2\pi i \lim_{N\to\infty} \frac{\log\langle L\rangle_N}{N} \pmod{\pi^2},$$

where $\mathrm{cs}(L)$ is the Chern-Simons invariant of the cusped manifold $S^3\backslash L$.

Introduction

Theorem (Yokota)

For a hyperbolic knot K,

$$\mathrm{vol}(K) = \mathrm{Im}\left\{ 2\pi i \operatorname*{o-lim}_{N\to\infty} \frac{\log\langle K\rangle_N}{N} \right\}.$$

Theorem (Yokota, to appear)

For a hyperbolic knot K,

$$i(\mathrm{vol}(K) + i\,\mathrm{cs}(K)) \equiv 2\pi i \operatorname*{o-lim}_{N\to\infty} \frac{\log\langle K\rangle_N}{N} \pmod{\pi^2}.$$

Introduction

Theorem (H. Murakami and J. Murakami(2001))

For a knot K,

$$\langle K \rangle_N = J_N\left(K; \exp(\frac{2\pi i}{N})\right)$$

where $J_N(K;u)$ is the N-th colored Jones polynomial of the knot K evaluated at $u \in \mathbb{C}$.

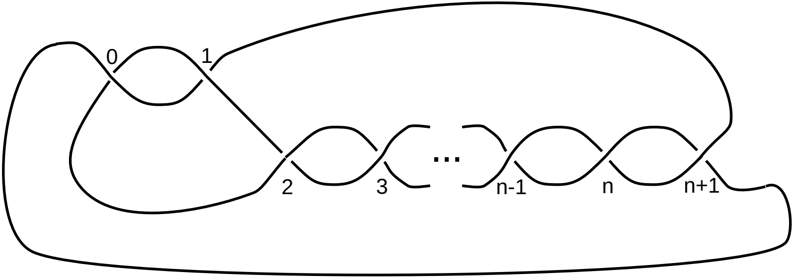
Theorem (Ohnuki(2005))

For a two-bridge knot K,

$$\mathrm{vol}(K) = -\mathrm{Im}\left\{2\pi i \operatorname*{o-lim}_{N\to\infty} \frac{\log J_N\left(K; \exp(\frac{2\pi i}{N})\right)}{N}\right\}.$$

Introduction

Let T_n $(n \geq 3)$ be the twist knot with $n+2$ crossings as below.



Main Theorem

Let $q := \exp(\frac{2\pi i}{N})$ for an integer $N \geq 2$.

Conjecture (Volume conjecture of colored Jones polynomial)

For a hyperbolic link L,

$$i(\mathrm{vol}(L) + i\mathrm{cs}(L)) \equiv -2\pi i \underset{N\to\infty}{\text{o-lim}} \frac{\log J_N(L;q)}{N} \pmod{\pi^2}.$$

Theorem (Cho and J. Murakami(2010))

For a twist knot T_n,

$$i(\mathrm{vol}(T_n) + i\mathrm{cs}(T_n)) \equiv -2\pi i \underset{N\to\infty}{\text{o-lim}} \frac{\log J_N(T_n;q)}{N} \pmod{\pi^2}.$$

Contents

1. Colored Jones polynomial

Definition of $\mathcal{U}_q(sl_2(\mathbb{C}))$

The quantum group $\mathcal{U}_q(sl_2(\mathbb{C}))$ is a finite dimensional algebra over $\mathbb{C}$ with generators X, Y, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1,\ KX = q^{1/2}XK,\ KY = q^{-1/2}YK,$$
$$XY - YX = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},\ X^N = Y^N = 0,\ K^{4N} = 1.$$

$\mathcal{U}_q(sl_2(\mathbb{C}))$ has a Hopf algebra structure by

$$\Delta(X) = X \otimes K + K^{-1} \otimes X,\ \Delta(Y) = Y \otimes K + K^{-1} \otimes Y,$$
$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$
$$S(X) = -q^{1/2}X,\ S(Y) = -q^{-1/2}Y,\ S(K^{\pm 1}) = K^{\mp 1},$$
$$\epsilon(X) = \epsilon(Y) = 0,\ \epsilon(K^{\pm 1}) = 1$$

where Δ is the comultiplication, S is the antipode and ϵ is the counit.

Representation of $\mathcal{U}_q(sl_2(\mathbb{C}))$

Let V^r be the r-dimensional vector space over $\mathbb{C}$ with basis $\{e_0, e_1, \ldots, e_{r-1}\}$ for $r \leq N$. Also let $[n] := \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}$ for a nonnegative integer n. Then we define the action of $\mathcal{U}_q(sl_2(\mathbb{C}))$ by

$$\begin{aligned} X \cdot e_j &= \begin{cases} [j+1]e_{j+1} & \text{if } j < r-1, \\ 0 & \text{if } j = r-1, \end{cases} \\ Y \cdot e_j &= \begin{cases} [j]e_{j-1} & \text{if } j > 0, \\ 0 & \text{if } j = 0, \end{cases} \\ K \cdot e_j &= q^{j/2-(r-1)/4} e_j. \end{aligned}$$

Therefore V^r becomes an irreducible $\mathcal{U}_q(sl_2(\mathbb{C}))$-module.
Because $\mathcal{U}_q(sl_2(\mathbb{C}))$ is a Hopf algebra, $(V^r)^*$ and $V^r \otimes V^{r'}$ also become $\mathcal{U}_q(sl_2(\mathbb{C}))$-modules. We put the dual basis of $(V^r)^*$ to be $\{e^0, e^1, \ldots, e^{r-1}\}$.

R-matrix

Let $[n]! := [n][n-1]\dots[1]$ and $[0]! := 1$. The element

$$R = \frac{1}{4N}\sum_{n,a,b}\frac{(q^{1/2}-q^{-1/2})}{[n]!}q^{-(ab-an+bn+n)/4}X^nK^a \otimes Y^nK^b$$

in $\mathcal{U}_q(sl_2(\mathbb{C})) \otimes \mathcal{U}_q(sl_2(\mathbb{C}))$, where the sum is over all $0 \le n < r$ and $0 \le a, b < 4r$, is the universal R-matrix for $\mathcal{U}_q(sl_2(\mathbb{C}))$.
For vectors $v \in V^r$ and $w \in V^{r'}$, we define module homomorphisms

$$P(v \otimes w) := w \otimes v,$$
$$\check{R}(v \otimes w) = P(R \cdot v \otimes w).$$

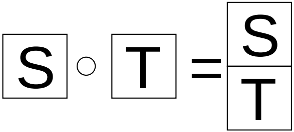
Then

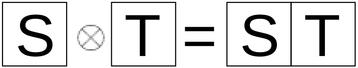
$$(\check{R} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R})(\check{R} \otimes \mathrm{id}) = (\mathrm{id} \otimes \check{R})(\check{R} \otimes \mathrm{id})(\mathrm{id} \otimes \check{R})$$

holds, i.e. Yang-Baxter equation holds.

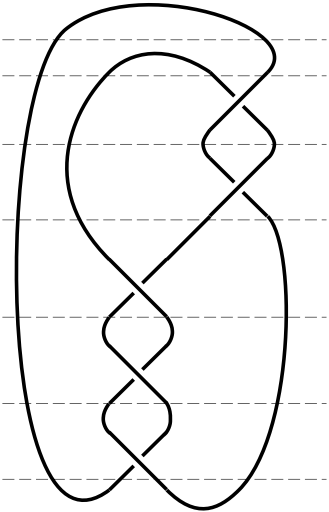
Elementary tangle diagrams with operations

I R L ∩ U





Example with 5_2 knot



$$\cap \circ$$
$$(| \otimes \cap \otimes |) \circ$$
$$(| \otimes | \otimes R) \circ$$
$$(| \otimes | \otimes R) \circ$$
$$(| \otimes L \otimes |) \circ$$
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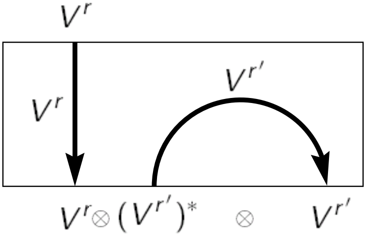
Operator invariant of tangle diagrams

Coloring of each arcs :



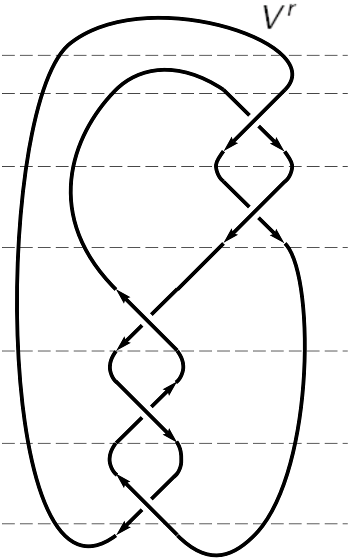
Assign the following functions to each elementary tangle diagram

$$| \longleftrightarrow \mathrm{id},\ R \longleftrightarrow \check{R},\ L \longleftrightarrow \check{R}^{-1}$$

$$\curvearrowright \longleftrightarrow E,\ \curvearrowleft \longleftrightarrow E_{K^2},\ \smile \longleftrightarrow N,\ \smile N_{K^{-2}}$$

where $E(f \otimes x) = f(x)$, $E_{K^2}(x \otimes f) = f(K^2 \cdot x)$, $N(1) = \sum_j e_j \otimes e^j$ and $N_{K^{-2}}(1) = \sum_j e_j \otimes (K^{-2} \cdot e_j)$.

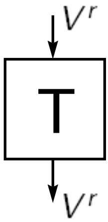
Example with 5_2 knot



$\bigcap \circ$	$E \circ$
$(\vert \otimes \bigcap \otimes \vert) \circ$	$(\mathrm{id} \otimes E \otimes \mathrm{id}) \circ$
$(\vert \otimes \vert \otimes R) \circ$	$(\mathrm{id} \otimes \mathrm{id} \otimes \check{R}) \circ$
$(\vert \otimes \vert \otimes R) \circ$	$(\mathrm{id} \otimes \mathrm{id} \otimes \check{R}) \circ$
$(\vert \otimes L \otimes \vert) \circ$	$(\mathrm{id} \otimes \check{R}^{-1} \otimes \mathrm{id}) \circ$
$(\vert \otimes L \otimes \vert) \circ$	$(\mathrm{id} \otimes \check{R}^{-1} \otimes \mathrm{id}) \circ$
$(\vert \otimes L \otimes \vert) \circ$	$(\mathrm{id} \otimes \check{R}^{-1} \otimes \mathrm{id}) \circ$
$(\bigcup \circ \bigcup)$	$(N_{K^{-2}} \otimes N_{K^{-2}}) \in \mathbb{C}$

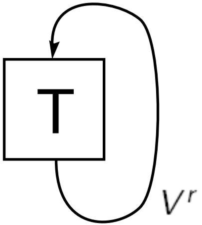
Colored Jones polynomial

Let T be a (1,1)-tangle with color V^r. Then, by Schur's Lemma



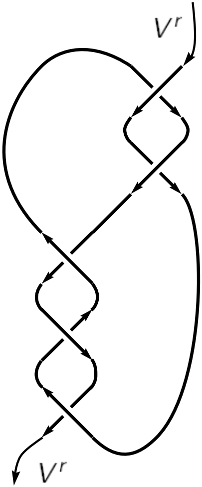
$\Longrightarrow c \cdot \mathrm{id}_{V^r}$

for some constant $c \in \mathbb{C}$, and



$\Longrightarrow c[r] \in \mathbb{C}$

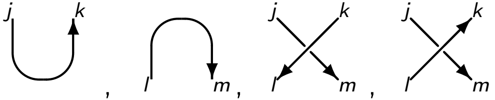
Colored Jones polynomial



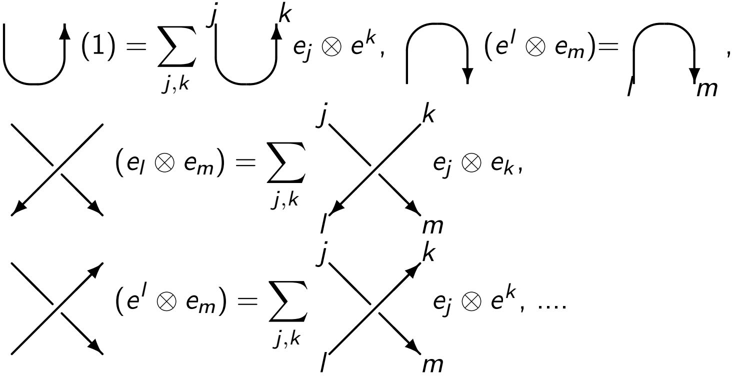
$$\Longrightarrow J_r(5_2; q) \cdot \mathrm{id}_{V^r}$$

For $2 \leq r \leq N$ and a (framed) link L, $J_r(L; q)$ is called the r-th colored Jones polynomial.

Calculation of colored Jones polynomial

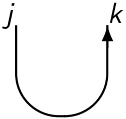
Let 

, ... be the complex numbers satisfying

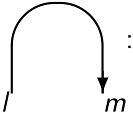


Calculation of colored Jones polynomial

Let $(q)_k := \prod_{m=1}^{k}(1 - q^m)$ for a positive integer k, and $(q)_0 := 1$. Then

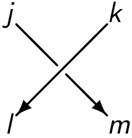


: $\delta_{j,k} q^{j-(N-1)/2}$,

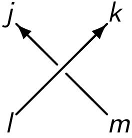


: $\delta_{l,m} q^{-l+(N-1)/2}$,

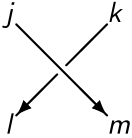
Calculation of colored Jones polynomial



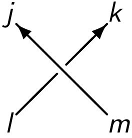
$$: \ \delta_{m,j+h}\delta_{l,k-h}\frac{(q)_j^{-1}(q)_k}{(q)_h(q)_l(q)_m^{-1}}(-1)^{j+l+1}q^{jl+(j+l)/2+(N^2+1)/4},$$



$$: \ \delta_{j,m+h}\delta_{k,l-h}\frac{(q)_j(q)_k^{-1}}{(q)_h(q)_l^{-1}(q)_m}(-1)^{k+m+1}q^{km+(k+m)/2+(N^2+1)/4},$$

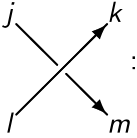


$$: \ \delta_{m,j-h}\delta_{l,k+h}\frac{(\overline{q})_j(\overline{q})_k^{-1}}{(\overline{q})_h(\overline{q})_l^{-1}(\overline{q})_m}(-1)^{k+m+1}q^{-km-(k+m)/2-(N^2+1)/4},$$

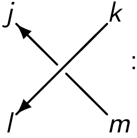


$$: \ \delta_{j,m-h}\delta_{k,l+h}\frac{(\overline{q})_j^{-1}(\overline{q})_k}{(\overline{q})_h(\overline{q})_l(\overline{q})_m^{-1}}(-1)^{j+l+1}q^{-jl-(j+l)/2-(N^2+1)/4},$$

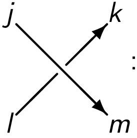
Calculation of colored Jones polynomial



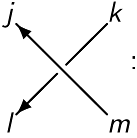
$$: \delta_{k,l-h}\delta_{m,j+h}\frac{(\overline{q})_j^{-1}(\overline{q})_k^{-1}}{(\overline{q})_h(\overline{q})_l^{-1}(\overline{q})_m^{-1}}(-1)^{j+k+1}q^{-jk-(l+m)/2-(N^2+1)/4},$$



$$: \delta_{l,k-h}\delta_{j,m+h}\frac{(\overline{q})_j(\overline{q})_k}{(\overline{q})_h(\overline{q})_l(\overline{q})_m}(-1)^{l+m+1}q^{-lm-(j+k)/2-(N^2+1)/4},$$



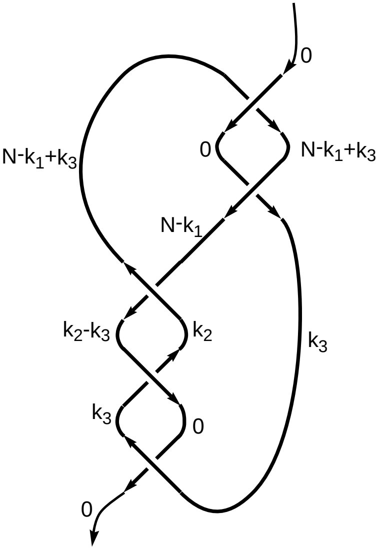
$$: \delta_{k,l+h}\delta_{m,j-h}\frac{(q)_j(q)_k}{(q)_h(q)_l(q)_m}(-1)^{l+m+1}q^{lm+(j+k)/2+(N^2+1)/4},$$



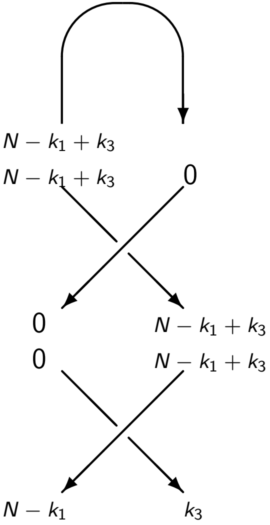
$$: \delta_{l,k+h}\delta_{j,m-h}\frac{(q)_j^{-1}(q)_k^{-1}}{(q)_h(q)_l^{-1}(q)_m^{-1}}(-1)^{j+k+1}q^{jk+(l+m)/2+(N^2+1)/4}.$$

Example of $J_N(5_2; q)$

Consider the non-degenerate state of the 5_2 knot diagram.



Example of $J_N(5_2; q)$

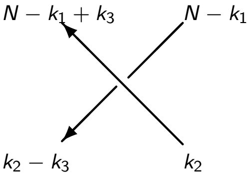


$: \ q^{k_1-k_3+\frac{N-1}{2}}$ (a)

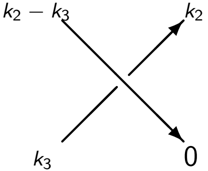
$: \ (-1)^{N-k_1+k_3+1} q^{\frac{N-k_1+k_3}{2}+\frac{N^2+1}{4}}$ (b)

$: \ \frac{(q)_{N-k_1+k_3}}{(q)_{N-k_1}}(-1)^{N-k_1+1} q^{\frac{N-k_1}{2}+\frac{N^2+1}{4}}$ (c)

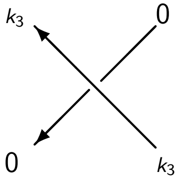
Example of $J_N(5_2; q)$



$$: \frac{(\overline{q})_{k_2-k_3}(q)_{k_2}}{(\overline{q})_{k_2+k_1-k_3-N}(q)_{N-k_1+k_3}(\overline{q})_{N-k_1}} \times(-1)^{k_3+1} q^{-k_1k_2+\frac{2k_2-k_3}{2}+\frac{N^2+1}{4}} \qquad \text{(d)}$$



$$: \frac{(q)_{k_2}}{(q)_{k_3}}(-1)^{k_3+1} q^{\frac{2k_2-k_3}{2}+\frac{N^2+1}{4}} \qquad \text{(e)}$$



$$: (-1)^{k_3+1} q^{\frac{k_3}{2}+\frac{N^2+1}{4}} \qquad \text{(f)}$$

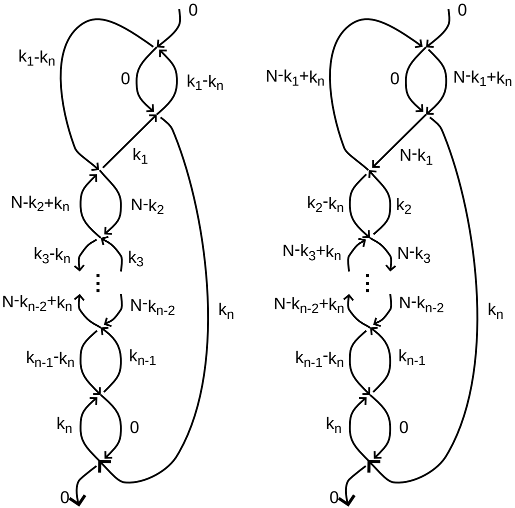
Example of $J_N(5_2; q)$

Therefore,

$$J_N(5_2; q) = \sum_{k_1, k_2, k_3} \Big\{ (a)(b)(c)(d)(e)(f) \Big\}.$$

$J_N(T_n; q)$

Similarily, we can calculate $J_N(T_n; q)$ using the following state.



(a) n is even (b) n is odd

2. Optimistic limit of colored Jones polynomial

Formal substitution

Consider the following formal substitution of the colored Jones polynomial

$$(q)_k \sim \exp\frac{N}{2\pi i}\left\{-\mathrm{Li}_2(q^k)+\frac{\pi^2}{6}\right\}, (\overline{q})_k \sim \exp\frac{N}{2\pi i}\left\{\mathrm{Li}_2(\overline{q}^k)-\frac{\pi^2}{6}\right\},$$

$$q^{kk'} \sim \exp\frac{N}{2\pi i}\left\{\log q^k+\log q^{k'}\right\},$$

and substitute q^{k_m} by w_m. In the case of $J_N(5_2;q)$,

$$\begin{aligned}
(a),(b),(f) &\sim 1,\\
(c) &\sim \exp\frac{N}{2\pi i}\left\{\mathrm{Li}_2(\frac{1}{w_1})-\mathrm{Li}_2(\frac{w_3}{w_1})\right\},\\
(d) &\sim \exp\frac{N}{2\pi i}\left\{\mathrm{Li}_2(\frac{w_3}{w_2})-\mathrm{Li}_2(w_2)-\mathrm{Li}_2(\frac{w_3}{w_1w_2})\right.\\
&\qquad \left.+\mathrm{Li}_2(\frac{w_3}{w_1})-\mathrm{Li}_2(w_1)+\frac{\pi^2}{6}-\log w_1\log w_2\right\},\\
(e) &\sim \exp\frac{N}{2\pi i}\{-\mathrm{Li}_2(w_2)+\mathrm{Li}_2(w_3).\}
\end{aligned}$$

Formal substitution

Therefore,

$$2\pi i\frac{\log J_N(5_2;q)}{N}\sim \mathrm{Li}_2(\frac{1}{w_1})-\mathrm{Li}_2(w_1)-2\mathrm{Li}_2(w_2)-\log w_1\log w_2+\frac{\pi^2}{6}$$
$$+\mathrm{Li}_2(w_3)+\mathrm{Li}_2(\frac{w_3}{w_2})-\mathrm{Li}_2(\frac{w_3}{w_1 w_2}).$$

Likewise, we define a function $W(T_n;w_1,w_2,\ldots,w_{n+1})$ by

$$2\pi i\frac{\log J_N(T_n;q)}{N}\sim W(T_n;w_1,w_2,\ldots,w_{n+1}).$$

Then we can obtain

$$W(T_n)=\mathrm{Li}_2(\frac{1}{w_1})-\mathrm{Li}_2(\frac{w_n}{w_1})$$
$$+\sum_{m=1}^{n-2}\Bigg(-\mathrm{Li}_2(w_m)-\mathrm{Li}_2(w_{m+1})-\log w_m\log w_{m+1}+\frac{\pi^2}{6}$$
$$+\mathrm{Li}_2(\frac{w_n}{w_m})+\mathrm{Li}_2(\frac{w_n}{w_{m+1}})-\mathrm{Li}_2(\frac{w_n}{w_m w_{m+1}})\Bigg)-\mathrm{Li}_2(w_{n-1})+\mathrm{Li}_2(w_n)\Bigg\}.$$

Optimistic limit

The optimistic limit $2\pi i \underset{N\to\infty}{\text{o-lim}} \frac{\log J_N(T_n;q)}{N}$ is defined as follows.
Choose some solution $(w_1, w_2, \ldots, w_{n-1}, w_n)$ of the set of equations

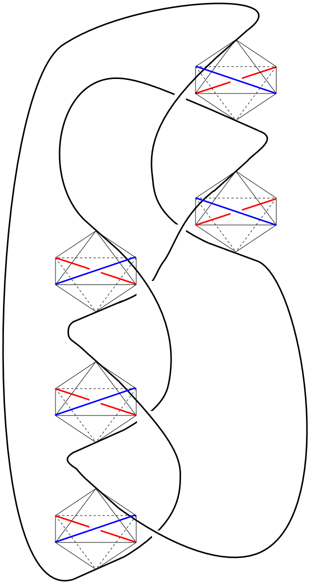
$$\left\{ \exp\left(w_m \frac{\partial W}{\partial w_m} \right) = 1 \,\middle|\, m = 1, 2, \ldots, n \right\}.$$

(We choose $w_n = 0$ here.) Then, the result of evaluating it to

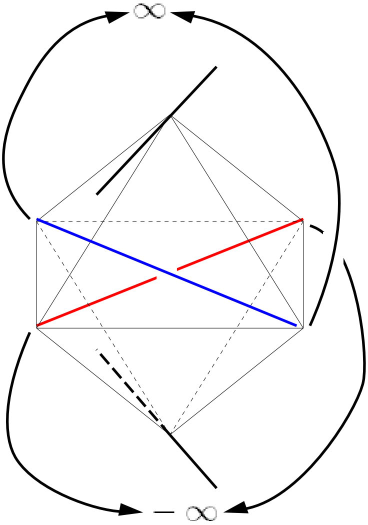
$$W(T_n; w_1, w_2, \ldots, w_{n-1}, 0) - \sum_{m=1}^{n-1} \left(w_m \frac{\partial W}{\partial w_m} \right) \log w_m$$

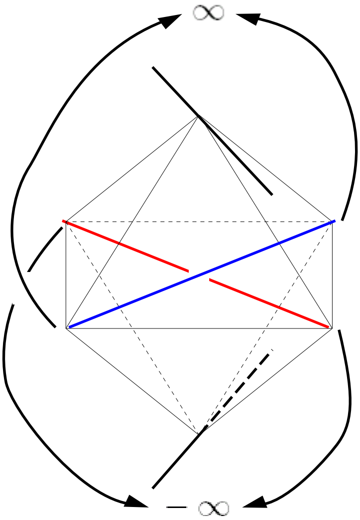
is called the optimistic limit.

Ideal triangulation

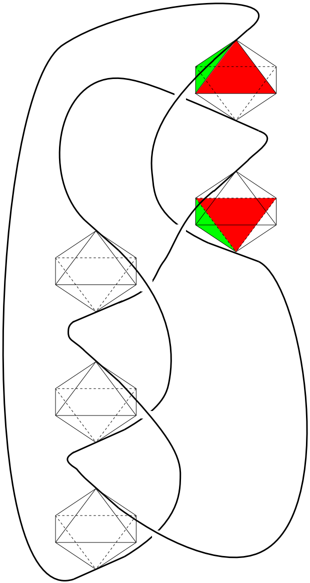


Ideal triangulation

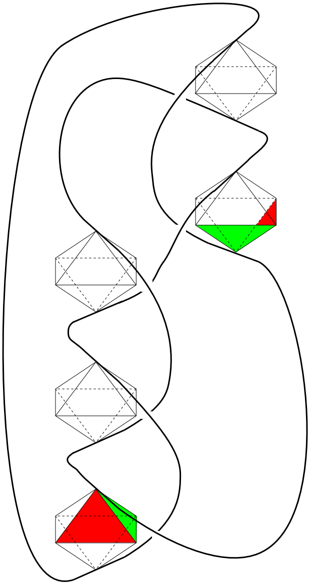




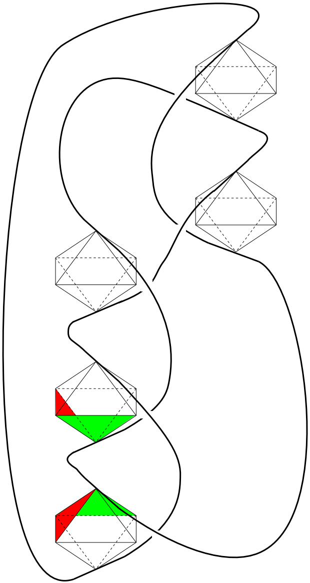
Ideal triangulation



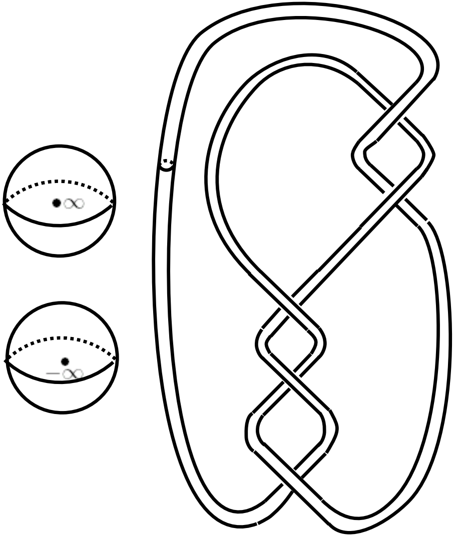
Ideal triangulation



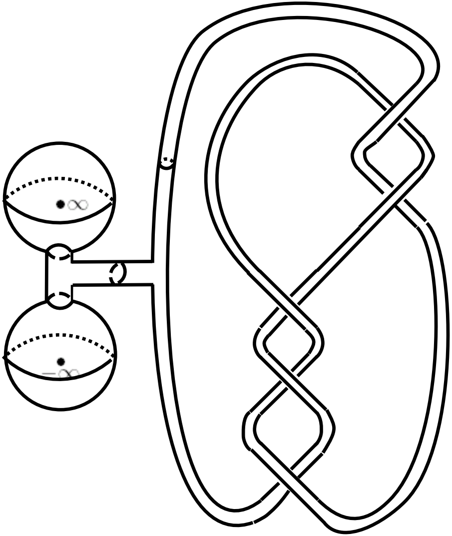
Ideal triangulation



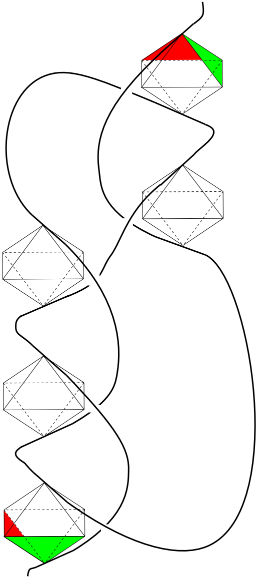
Ideal triangulation



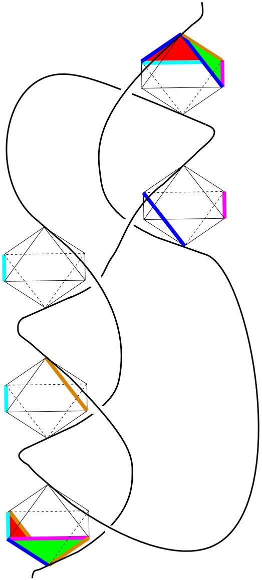
Ideal triangulation



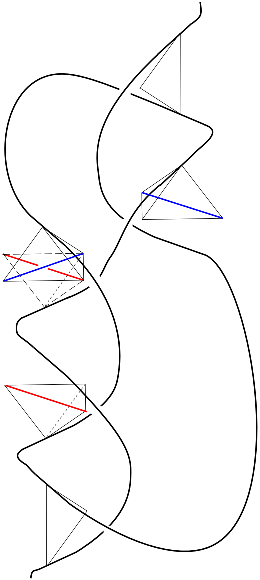
Ideal triangulation



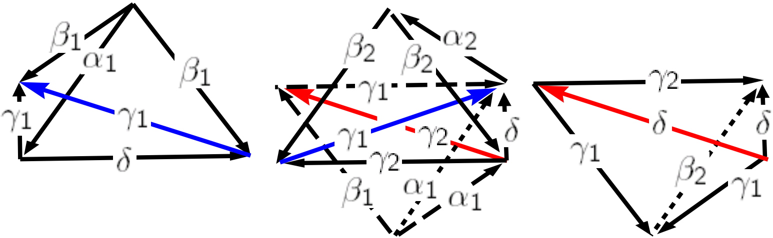
Ideal triangulation



Ideal triangulation

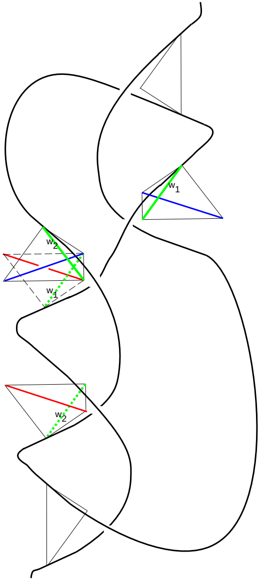


Ideal triangulation



(Note that $\gamma_1 = \alpha_2$.)

Parametrization



Hyperbolicity equations

The set of the hyperbolicity equations of this triangulation is

$$\left\{ \frac{w_2}{(1-w_1)(1-\frac{1}{w_1})} = 1,\ \frac{w_1}{(1-w_2)^2} = 1 \right\},$$

which coincides with

$$\left\{ \exp\left(w_1 \frac{\partial W(5_2; w_1, w_2, 0)}{\partial w_1} \right) = 1, \exp\left(w_2 \frac{\partial W(5_2; w_1, w_2, 0)}{\partial w_2} \right) = 1 \right\}.$$

Hyperbolicity equations

In general, the set of the hyperbolicity equations of T_n becomes

$$\left\{\begin{aligned} \frac{w_2}{(1-w_1)(1-\frac{1}{w_1})} &= 1, \\ \frac{w_{m-1}w_{m+1}}{(1-w_m)^2} &= 1, \quad \text{for } m = 2, 3, \ldots, n-2, \\ \frac{w_{n-2}}{(1-w_{n-1})^2} &= 1 \end{aligned}\right\}$$

which coincides with

$$\left\{ \exp\left(w_m \frac{\partial W(T_n; w_1, w_2, \ldots, w_{n-1}, 0)}{\partial w_m} \right) = 1 \,\middle|\, m = 1, 2, \ldots, n-1 \right\}.$$

On the other hands, the existence of *the geometric solution*, which gives the hyperbolic structure of the knot complement with $0 < \operatorname{Im} w_m < \pi$ for $m = 1, 2, \ldots, n-1$, was already proved by Hoste and Shanahan(2001).

Main theorem

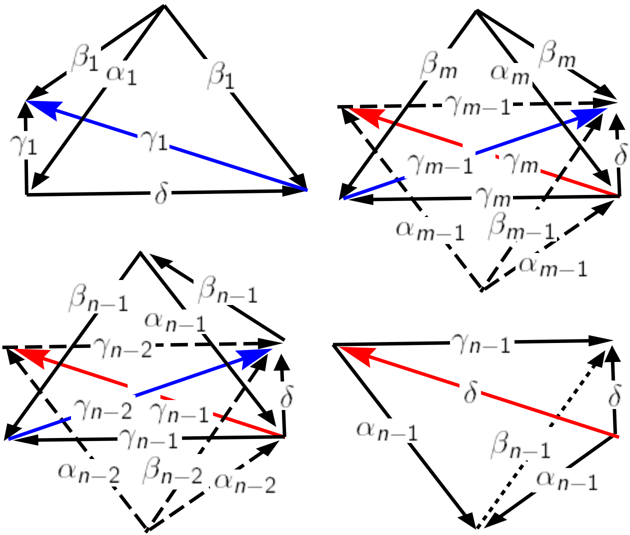
Theorem (Cho and J. Murakami(2010))

Let $(w_1, w_2, \ldots, w_{n-1})$ *be the geometric solution of* T_n*. Then*

$$\begin{aligned} & 2\pi i \ \underset{N\to\infty}{\text{o-lim}} \frac{\log J_N\,(T_n; q)}{N} \\ := \quad & W(T_n; w_1, w_2, \ldots, w_{n-1}, 0) - \sum_{m=1}^{n-1} \left(w_m \frac{\partial W}{\partial w_m} \right) \log w_m \\ \equiv \quad & -i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \pmod{\pi^2}. \end{aligned}$$

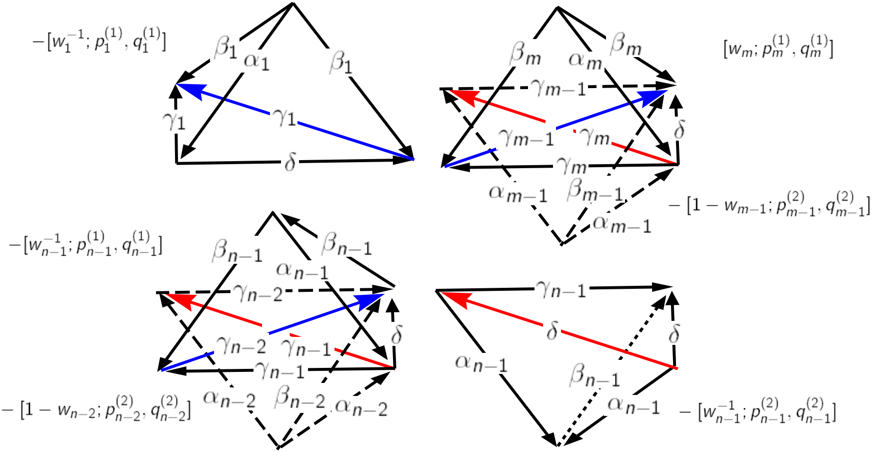
3. Complex volume of T_n

Elements of extended Bloch group



$(m = 2, 3, \ldots, n-2.$ Note that $\gamma_1 = \alpha_{n-1}.)$

Elements of extended Bloch group



$$(m = 2, 3, \ldots, n-2)$$

Elements of extended Bloch group

$$p_m^{(1)}\pi i = \begin{cases} \log w_1 + \log\delta - \log\gamma_1 & \text{for } m = 1, \\ -\log w_m + \log\gamma_m - \log\delta & \text{for } m = 2, 3, \ldots, n-2, \\ \log w_{n-1} + \log\delta - \log\gamma_{n-1} & \text{for } m = n-1, \end{cases}$$

$$q_m^{(1)}\pi i = \begin{cases} \log(1 - \frac{1}{w_1}) + \log\beta_1 - \log\alpha_1 & \text{for } m = 1, \\ \log(1 - w_m) + \log\beta_m + \log\delta - \log\alpha_m - \log\gamma_{m-1} & \\ \qquad \text{for } m = 2, 3, \ldots, n-2, & \\ \log(1 - \frac{1}{w_{n-1}}) + \log\gamma_{n-1} + \log\beta_{n-1} - \log\alpha_{n-1} - \log\gamma_{n-2} & \\ \qquad \text{for } m = n-1, & \end{cases}$$

$$p_m^{(2)}\pi i = \begin{cases} -\log(1 - w_m) + \log\beta_m + \log\gamma_{m+1} - \log\alpha_m - \log\delta & \\ \qquad \text{for } m = 1, 2, \ldots, n-2, & \\ \log w_{n-1} + \log\delta - \log\gamma_{n-1} & \text{for } m = n-1, \end{cases}$$

$$q_m^{(2)}\pi i = \begin{cases} \log w_m + \log\delta - \log\gamma_m & \text{for } m = 1, 2, \ldots, n-2, \\ \log(1 - \frac{1}{w_{n-1}}) + \log\alpha_{n-1} + \log\gamma_{n-1} - \log\delta - \log\beta_{n-1} & \\ \qquad \text{for } m = n-1. & \end{cases}$$

Complex volume

Evaluating $\sum[w;p,q]$ to
$\widehat{L}[w;p,q] = \mathrm{Li}_2(w) + \frac{1}{2}\log w \log(1-w) + \frac{\pi i}{2}(q \log w + p \log(1-w)) - \frac{\pi^2}{6}$,
we obtain

$$\begin{aligned} i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) &\equiv \widehat{L}\left(\sum[w;p,q]\right) \\ &\equiv -W(T_n; w_1, w_2, \ldots, w_{n-1}, 0) \\ &\quad + \sum_{m=1}^{n-1}\left(w_m \frac{\partial W}{\partial w_m}\right)\log w_m \pmod{\pi^2}. \end{aligned}$$

4. Application to Yokota theory

Yokota theory

Yokota defined some triangulation of T_n and some function $V(z_1, z_2, \ldots, z_{n-1})$ from the formal substitution of Kashaev's invariant by

$$V := \mathrm{Li}_2\left(\frac{1}{z_1}\right) + \sum_{k=2}^{n-1}\left\{\frac{\pi^2}{6} - \mathrm{Li}_2(z_{k-1}) + \mathrm{Li}_2\left(\frac{z_{k-1}}{z_k}\right) - \mathrm{Li}_2\left(\frac{1}{z_k}\right)\right\} - \mathrm{Li}_2(z_{n-1}).$$

Let $(z_1, z_2, \ldots, z_{n-1})$ be the geometric solution corresponding to Yokota triangulation. Then the following holds.

$$z_1 = \left(1 - \frac{1}{w_1}\right)^{-1}, \ z_{m-1} = \frac{1 - w_{m-1}}{w_m}, \ z_m = \frac{w_{m-1}}{1 - w_m},$$

$$\frac{z_{k-1}}{z_k} = \left(1 - \frac{1}{w_{m-1}}\right)\left(1 - \frac{1}{w_m}\right), \ z_{n-1} = 1 - w_{n-1}$$

and

$$i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \equiv V(T_n; z_1, z_2, \ldots, z_{n-1}) - \sum_{m=1}^{n-1}\left(z_m \frac{\partial V}{\partial z_m}\right)\log z_m \pmod{\pi^2}.$$

Application to Yokota theory

On the other hands, we know the well-known dilogarithm identity

$$\mathrm{Li}_2\left(\frac{z_{m-1}}{z_m}\right) = \mathrm{Li}_2\left(z_{m-1}\right) + \mathrm{Li}_2\left(z_m^{-1}\right)$$
$$-\mathrm{Li}_2(1-w_{m-1}) - \mathrm{Li}_2(1-w_m) - \log w_{m-1} \log w_m.$$

Also we can find that, for the geometric solution $(w_1, w_2, \ldots, w_{n-1})$,

$$\left(w_m \frac{\partial W}{\partial w_m}\right) = \begin{cases} 0 & \text{for } k = 1, \\ 2\pi i & \text{for } k = 2, 3, \ldots, n-1, \end{cases}$$

using some edge relations. From these facts, we obtain

$$-W(T_n; w_1, w_2, \ldots, w_{n-1}, 0) + \sum_{m=1}^{n-1} \left(w_m \frac{\partial W}{\partial w_m}\right) \log w_m$$
$$\equiv V(z_1, z_2, \ldots, z_{n-1}) \pmod{\pi^2}.$$

Application to Yokota theory

As a result, we obtain the following corollary.

Corollary

$$\sum_{m=1}^{n-1} \left(z_m \frac{\partial V}{\partial z_m} \right) \log z_m \equiv 0 \pmod{\pi^2}$$

and

$$i(\mathrm{vol}(T_n) + i\,\mathrm{cs}(T_n)) \equiv V(T_n; z_1, z_2, \ldots, z_{n-1}) \pmod{\pi^2}.$$

Thank you very much for listening!