Preliminaries. The 3-dimensional hyperbolic space $\mathbb{H}^3$ is the upper half of $\mathbb{R}^3$ endowed with the metric

$$ds^2 = (dx^2 + dy^2 + dz^2)/z^2,$$

where $\partial\mathbb{H}^3$ is identified with $\mathbb{C}$. The group of orientation preserving isometries of $\mathbb{H}^3$ is PSL(2,$\mathbb{C}$), which naturally acts $\partial\mathbb{H}^3$. In what follows, by $T(z)$, we denote the ideal tetrahedron in $\mathbb{H}^3$ whose vertices are $0, 1, \infty$ and $z \in \mathbb{C}$. The volume of $T(z)$ is given by

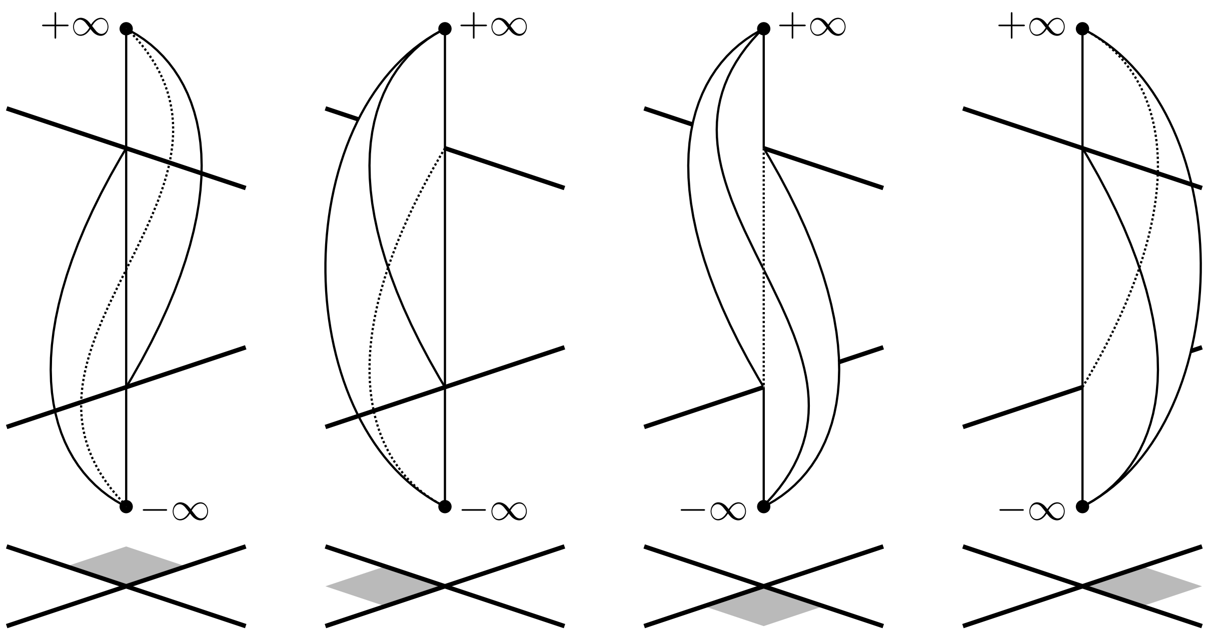
$$D(z) = \operatorname{Im}\operatorname{Li}_2(z) + \log|z|\arg(1-z).$$

Definition. Let K be a knot in S^3 and M its complement. We call K *hyperbolic* if there is a discrete, torsion-free subgroup Γ of PSL(2, $\mathbb{C}$) such that

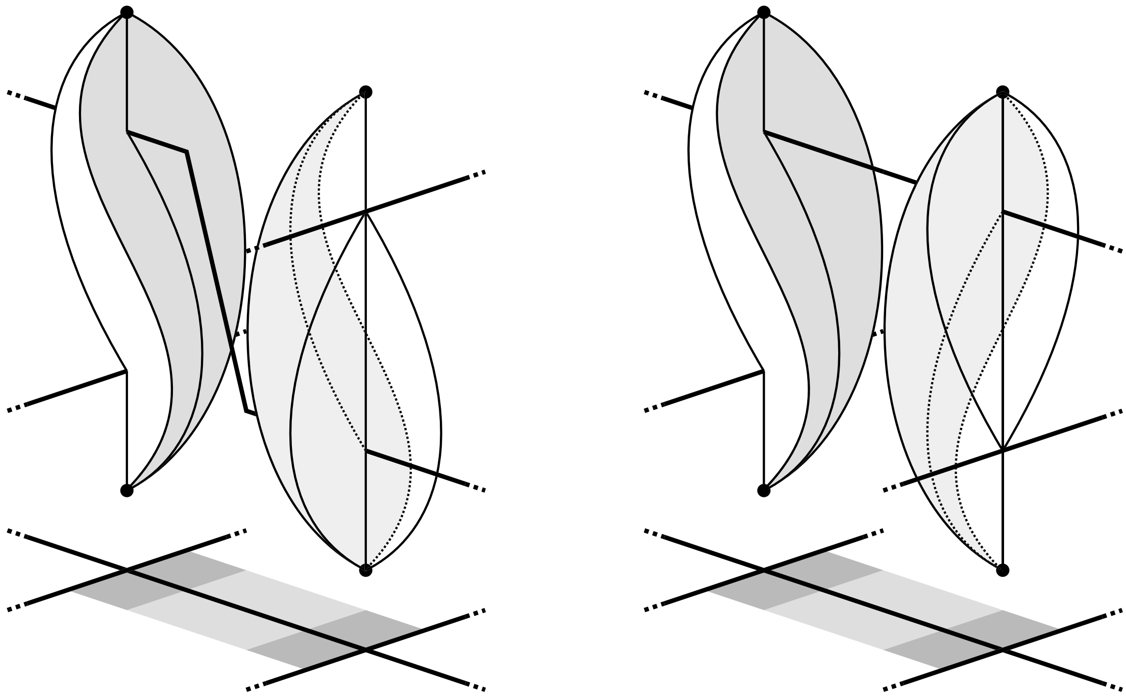
$$M = \mathbb{H}^3/\Gamma,$$

where Γ is called a *holonomy representation* of $\pi_1(M)$.

An ideal triangulation of $\dot{M}$. Let D be a diagram of K, and prepare 4 ideal tetrahedra at each crossing of D, where $\pm\infty$ denote the poles of S^3.



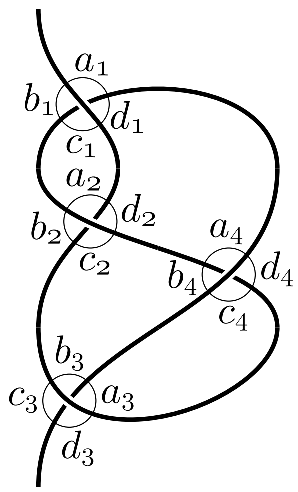
We glue them along the edges of D as follows.



Then, we obtain an ideal triangulation of

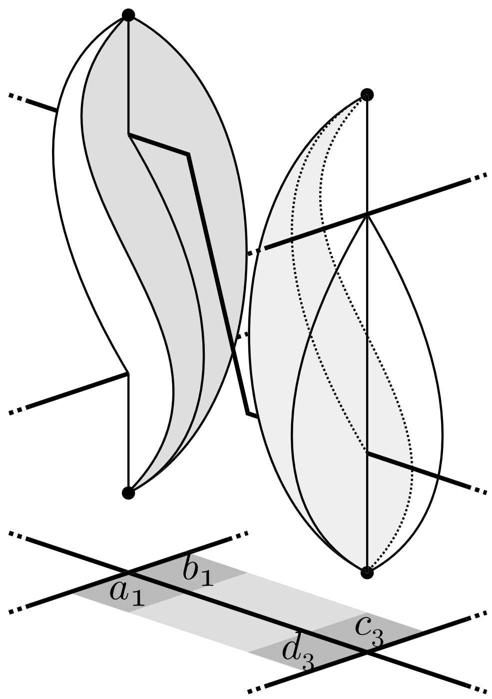
$$\dot{M} = M \setminus \{\pm\infty\}.$$

An ideal triangulation of M. Let us assign complex numbers to the corners of D and identify $T(z)$ with the tetrahedron corresponding to the corner assigned z.



In what follows, we suppose K is 4_1 and put

$$B = \{T(a_1) \cup T(b_1)\} \cap \{T(c_3) \cup T(d_3)\}.$$



As $\dot{M} \setminus B$ is homeomorphic to M, we can develop $\dot{M} \setminus B$ in $\mathbb{H}^3$, where the tetrahedra touching B can not specify distinct 4 points in $\partial\mathbb{H}^3$ and so degenerate. In fact,

$$T(a_1), T(b_1), T(c_3), T(d_3)$$

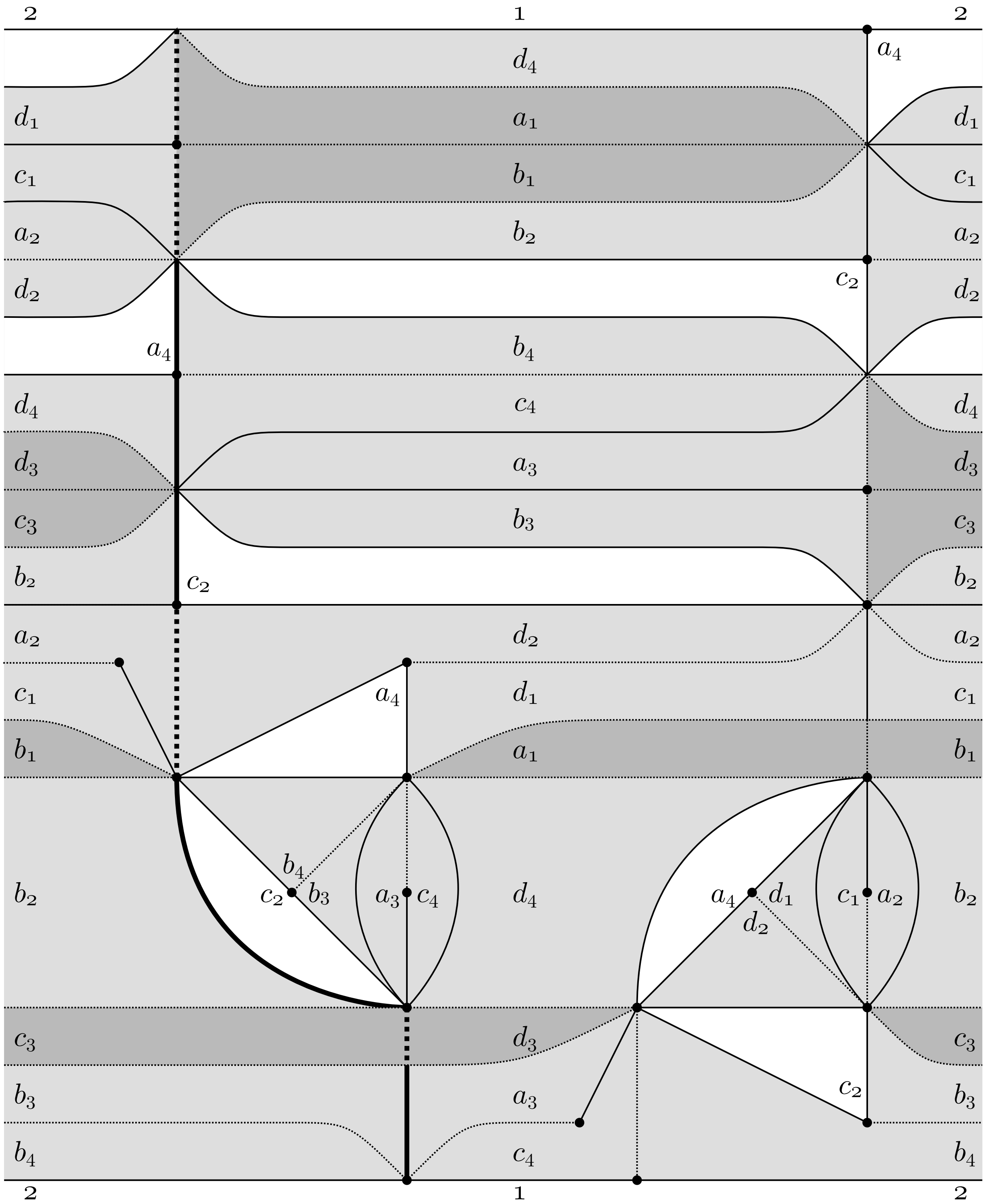
are essentially *one*-dimensional objects and

$$T(c_1), T(d_1), T(a_2), T(b_2), T(d_2), T(a_3), T(b_3),$$
$$T(b_4), T(c_4), T(d_4)$$

are essentially *two*-dimensional objects in $\dot{M} \setminus B$. Thus, we obtain an ideal triangulation $\mathcal{S}$ of M with

$$T(c_2), T(a_4).$$

A picture of $\partial N(B \cup K)$. Dotted edges are contracted.



Notice that any holizontal line represents a meridian of K and the bold curve represents the prefered longitude of K.

Hyperbolicity equations. Hyperbolicity equations for M can be read from the picture above, that is,

$$c_2/a_4 = (1 - 1/c_2)(1 - a_4) = m^2,$$

where m denotes the eigenvalue of the meridian in Γ. If we put $c_2 = xm, a_4 = x/m$, the equations become

$$(1) \qquad (1 - 1/xm)(1 - x/m) = m^2$$

and the hyperbolic structure of M corresponding to m is determined by a solution to (1). On the other hand, the eigenvalue l of the longitude in Γ is given by

$$l^2 = \frac{1 - 1/c_2}{c_2 a_4 (1 - a_4)} = \frac{1 - 1/xm}{x^2(1 - x/m)}.$$

In what follows, M_m denotes M with the hyperbolic structure obtained above.

Colored Jones polynomial. Due to Habiro and Le, the N-colored Jones polynomial $J_N(K, t)$ of K is given by

$$\sum_{n=0}^{N-1} \prod_{k=1}^{n} t^N (1 - t^{-N-k})(1 - t^{-N+k}).$$

From now on, we fix $r \in \mathbb{C} \setminus \mathbb{Q}$ near 1 and put

$$\omega = \exp \frac{2\pi i}{N}, \ q = \exp \frac{2\pi r i}{N}, \ m = \exp \pi(r - 1)i.$$

Asymptotics of q-factorials. Since

$$\prod_{k=1}^{n}(1-q^{-N\pm k})$$

is written as

$$\begin{aligned}
&\exp\frac{N}{2\pi}\left\{\frac{2\pi}{N}\sum_{k=1}^{n}\log(1-e^{\pm\frac{2\pi kri}{N}}/m^2)\right\}\\
=&\chi_\pm(n)\cdot\exp\frac{N}{2\pi}\left\{\int_0^{\frac{2\pi n}{N}}\log(1-e^{\pm tri}/m^2)\,dt\right\}\\
=&\chi_\pm(n)\cdot\exp\frac{N}{2\pi ri}\left\{\pm\int_{1/m^2}^{q^{\pm n}/m^2}\frac{\log(1-u)}{u}\,du\right\}\\
=&\chi_\pm(n)\cdot\exp\frac{N}{2\pi ri}\{\pm\mathrm{Li}_2(q^{\pm n}/m^2)\mp\mathrm{Li}_2(1/m^2)\},
\end{aligned}$$

we have

$$J_N(K,q)=\sum_{n=0}^{N-1}\chi(n)\cdot\exp\left\{\frac{N}{2\pi r\sqrt{-1}}\cdot H(q^n,m)\right\},$$

where

$$H(z,m)=\mathrm{Li}_2(1/zm^2)-\mathrm{Li}_2(z/m^2)+2\log z\log m.$$

Note that $\arg\chi(n)$ is bounded by a constant independent of N and $|\chi(n)|$ is bounded by a linear function of N and its inverse. Let $f_N(z)$ denote a complex function such that

$$f_N(\omega^n)=\chi(n).$$

Saddle point method. Then, by Cauchy's theorem, $J_N(K,q)$ is equal to

$$\frac{1}{2\pi i}\int_{C_+\cup C_-}\sum_{n=1}^{N-1}\frac{f_N(z)}{z-\omega^n}\exp\left\{\frac{N}{2\pi ri}\cdot H(z^r,m)\right\}dz$$

$$=\frac{N}{2\pi i}\int_{C_+\cup C_-}\frac{f_N(z)}{z(1-z^{-N})}\exp\left\{\frac{N}{2\pi ri}\cdot H(z^r,m)\right\}dz$$

$$=\frac{N}{2\pi i}\int_{C_+}\frac{f_N(z)}{z(1-z^{-N})}\exp\left\{\frac{N}{2\pi ri}\cdot H(z^r,m)\right\}dz$$

$$+\frac{N}{2\pi i}\int_{C_-}\frac{f_N(z)}{z(z^N-1)}\exp\frac{N}{2\pi ri}\left\{H(z^r,m)+2\pi i\log z^r\right\}dz,$$

where C_+ and C_- are obtained by pushing the curve

$$\left\{e^{i\theta}\in\mathbb{C}\;\middle|\;\frac{\pi}{N}\le\theta\le\frac{(2N-1)\pi}{N}\right\}$$

out and into the unit disk of $\mathbb{C}$. Then, by the saddle point method, we have

$$\int_{C_+}\frac{f_N(z)}{z(1-z^{-N})}\exp\left\{\frac{N}{2\pi ri}\cdot H(z^r,m)\right\}dz$$

is approximated by

$$\exp\left\{\frac{N}{2\pi ri}\cdot H(y,m)\right\}$$

when N goes to infinity, where y is a solution to

$$y^2-(m^2-1+1/m^2)y+1=0. \tag{2}$$

On the other hand, we can observe

$$\operatorname{Im}\{H(z^r, m) + 2\pi\sqrt{-1}\log z^r\} < \operatorname{Im} H(y, m)$$

for $z \in C_-$, and so we have

$$J_N(K, q) \sim \exp\left\{\frac{N}{2\pi r i} \cdot H(y, m)\right\}.$$

Neumann-Zagier function. From (1) and (2), we can observe

$$xm + ym^2 = 1, \quad m/x + m^2/y = 1.$$

In particular,

$$(1 - 1/xm)(1 - 1/ym^2) = 1, \quad (1 - x/m)(1 - y/m^2) = 1$$

and so we have

$$\begin{aligned} l^{-2} &= \frac{1 - x/m}{1 - 1/xm} \cdot x^2 = \left(\frac{1 - 1/ym^2}{1 - y/m^2} \cdot y\right)^2 \\ &= \exp\left\{m\frac{dH(y, m)}{dm}\right\}, \end{aligned}$$

which shows that $H(y, m)$ is nothing but the Neumann-Zagier function on the deformation space of M.

Volumes. Furthermore, $\operatorname{Im} H(y,m)$ is given by

$$\begin{aligned}
&D(1/ym^2) - D(y/m^2)\\
&+\log|y|\{\arg(1-1/ym^2)+\arg(1-y/m^2)+2\arg m\}\\
&+\log|m|\{2\arg(1-1/ym^2)-2\arg(1-y/m^2)+2\arg y\}\\
&=D(1/ym^2) - D(y/m^2) + \log|m|\cdot \operatorname{Im}\left\{m\frac{dH(y,m)}{dm}\right\},
\end{aligned}$$

where $D(1/ym^2) - D(y/m^2)$ is equal to

$$-D(1/xm) + D(x/m) = \operatorname{vol}(M_m).$$

Therefore we have

$$\operatorname{vol}(M_m) = \operatorname{Im} H(y,m) - \log|m|\cdot \operatorname{Im}\left\{m\frac{dH(y,m)}{dm}\right\}$$

for any $m \in \mathbb{C}\setminus\mathbb{Q}$ near 1, that is, the volume of M_m is determined by the function $H(y,m)$ and so determined by the colored Jones polynomials.