On an ideal triangulation and the A-polynomial of a knot

Naoko Tamura
Department of Mathematics, Tokyo Metropolitan University

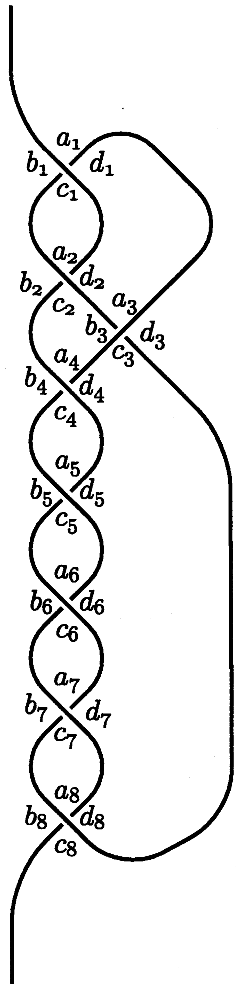

Figure 1: a diagram of a knot K

The examples of the A-polynomials.

$$A_{3_1} = 1 + lm^6, \qquad A_{4_1} = -m^4 + l(m^8 - m^6 - 2m^4 - m^2 + 1) - l^2m^4.$$

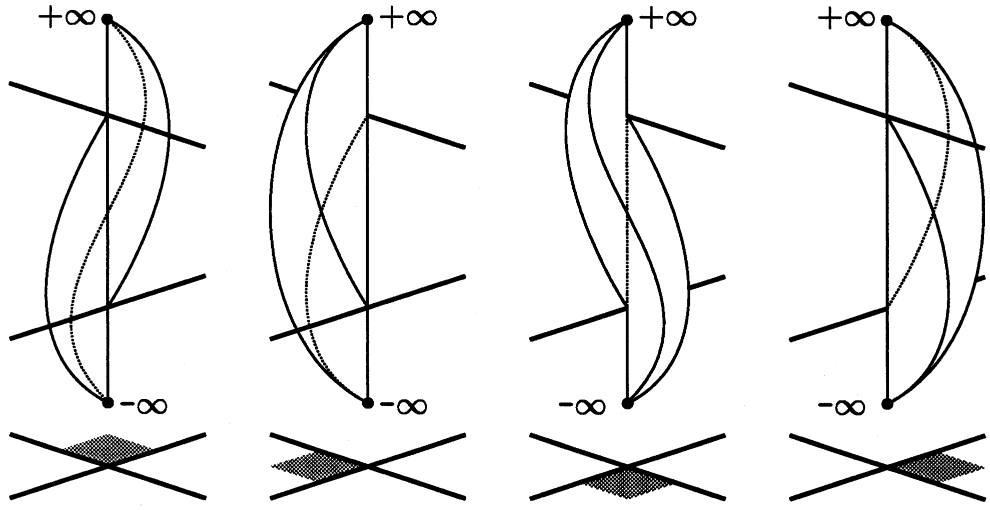

Figure 2:

The hyperbolicity equations about meridian are

$$
\begin{aligned}
m &= \frac{1-1/d_4}{a_4(1-d_5)} = \frac{1-1/d_6}{a_6} = \frac{(1-a_3)(1-1/c_2)}{1-1/b_3} \\
&= \frac{(1-1/a_4)(1-c_3)}{(1-d_4)(1-1/b_3)} = \frac{1-1/d_5}{a_5(1-d_6)},
\end{aligned}
$$

and about longitude is

$$
l = \frac{a_7c_3d_4d_5d_6a_6a_5a_4c_2}{a_3a_4a_5a_6a_7c_3d_4d_5d_6} \cdot \frac{1-1/c_2}{1-a_4} \cdot \frac{1}{c_2} \cdot m^6.
$$

The product of the moduli around each edge should be 1, so

$$
1 = a_3b_3c_3 = a_4c_4d_4 = a_5c_5d_5 = a_6c_6d_6 \tag{1}
$$

and

$$
1 = c_2b_3a_4 = c_4a_5 = c_5a_6 = c_6a_7 \tag{2}
$$

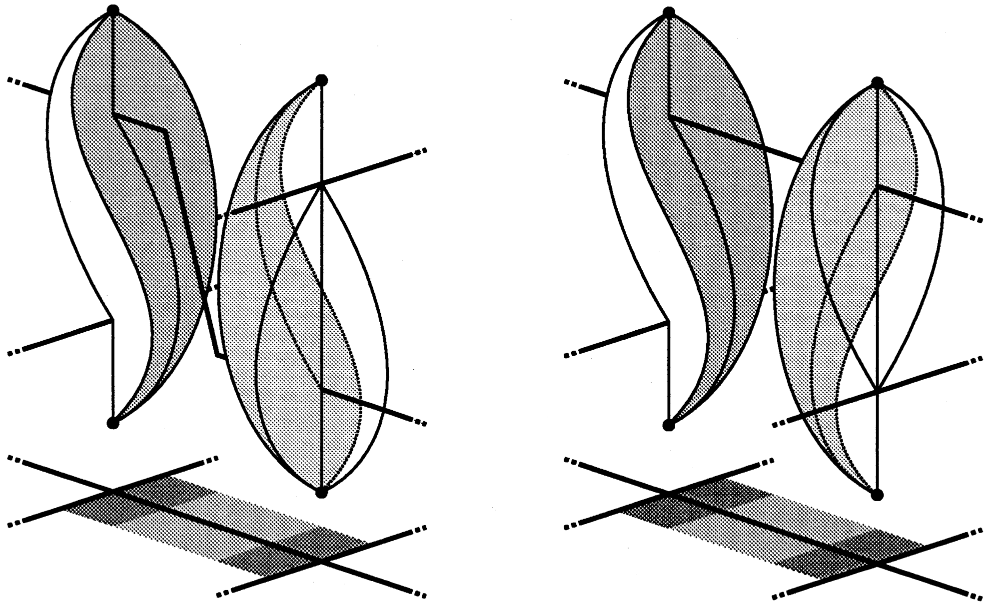

Figure 3:

From (1) and (2) we can put

$$c_2 = am, a_3 = a/m, b_3 = b/a, c_3 = m/b, a_4 = 1/bm, c_4 = cm, d_4 = b/c,$$
$$a_5 = 1/cm, c_5 = dm, d_5 = c/d, a_6 = 1/dm, c_6 = em, d_6 = d/e, a_7 = 1/em.$$

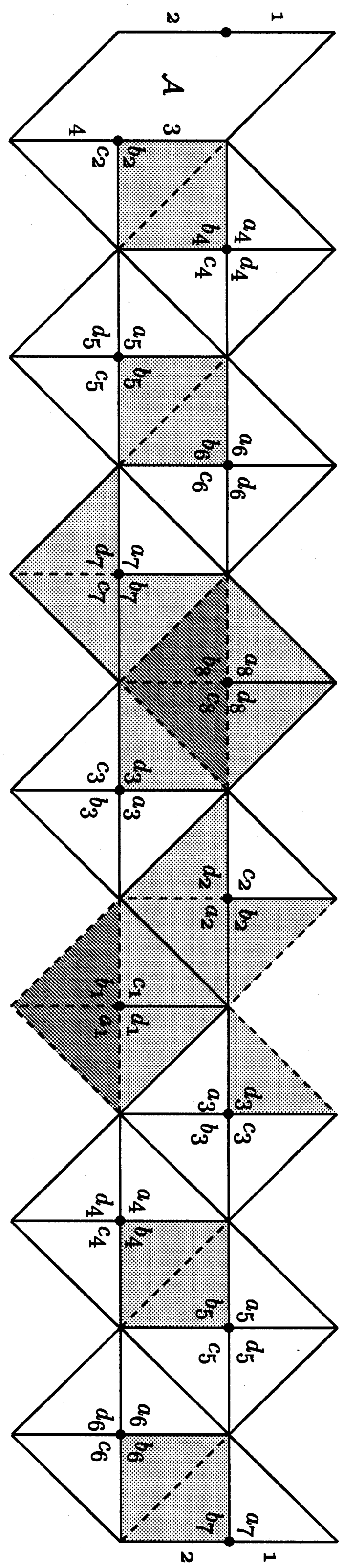

Figure 4: the triangulation of $\partial N(K)$

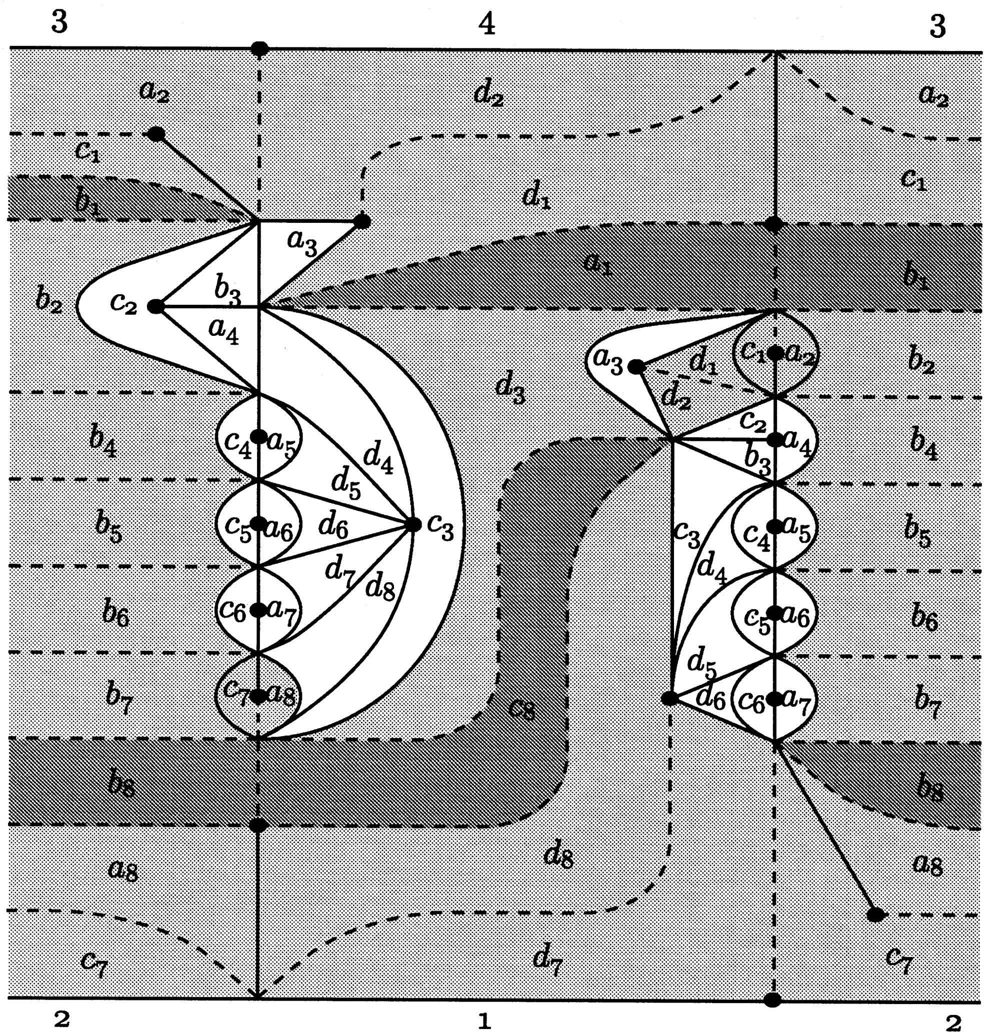

Figure 5: the triangulation of the annulus $\mathcal{A}$

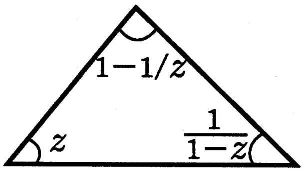

Figure 6:

Then we have the hyperbolicity equations for K.

$$
\begin{aligned}
m^2 &= \frac{bm(1-c/b)}{1-c/d} = dm(1-e/d) \\
&= \frac{(1-a/m)(1-1/am)}{1-a/b} = \frac{(1-bm)(1-m/b)}{(1-b/c)(1-a/b)} = \frac{cm(1-d/c)}{1-d/e},
\end{aligned}
$$

and

$$
l = \frac{m^7(1-1/am)}{a(1-1/bm)}.
$$

By eliminating the valiables except l, m we have the A-polynomial of K.

$$
\begin{aligned}
A_K(l,m) &= m^{72} + (-5m^{60} + 9m^{62} + 7m^{64} - 3m^{66} - m^{72} + 2m^{74} - m^{76})l \\
&+(10m^{48} - 32m^{50} - m^{52} + 56m^{54} + 17m^{56} - 28m^{58} - m^{60} + 14m^{62} \\
&\quad - 4m^{64} - 8m^{66} + 7m^{68} - 2m^{70})l^2 \\
&+(-10m^{36} + 42m^{38} - 24m^{40} - 87m^{42} + 29m^{44} + 143m^{46} + 33m^{48} \\
&\quad - 77m^{50} - 17m^{52} + 29m^{54} - 2m^{56} + m^{58} - 8m^{60} + 5m^{62} - m^{64})l^3 \\
&+(5m^{24} - 24m^{26} + 26m^{28} + 36m^{30} - 43m^{32} - 108m^{34} + 47m^{36} + 192m^{38} \\
&\quad + 47m^{40} - 108m^{42} - 43m^{44} + 36m^{46} + 26m^{48} - 24m^{50} + 5m^{52})l^4 \\
&+(-m^{12} + 5m^{14} - 8m^{16} + m^{18} - 2m^{20} + 29m^{22} - 17m^{24} - 77m^{26} \\
&\quad + 33m^{28} + 143m^{30} + 29m^{32} - 87m^{34} - 24m^{36} + 42m^{38} - 10m^{40})l^5 \\
&+(-2m^6 + 7m^8 - 8m^{10} - 4m^{12} + 14m^{14} - m^{16} - 28m^{18} + 17m^{20} \\
&\quad + 56m^{22} - m^{24} - 32m^{26} + 10m^{28})l^6 \\
&+(-1 + 2m^2 - m^4 - 3m^{10} + 7m^{12} + 9m^{14} - 5m^{16})l^7 + m^4 l^8
\end{aligned}
$$

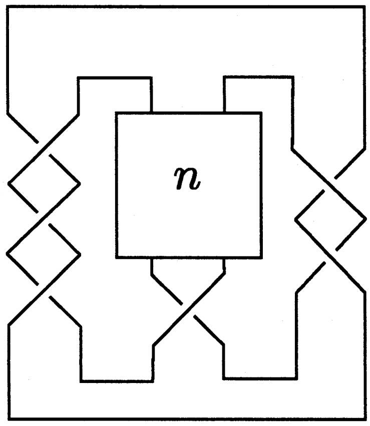

Figure 7: K_n

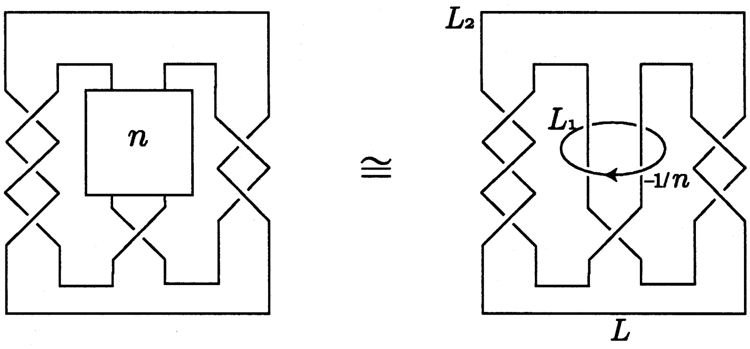

Figure 8: K_n and L

$$A_{K_0}(l,m) = 1+lm^{10},\ A_{K_1}(lm^{-4},m) = 1+lm^{8},\ A_{K_2}(lm^{-8},m) = (1+lm^{7})(1-lm^{7}),$$

$$\begin{aligned} A_{K_3}(lm^{-12},m) = 1 + (-m^4+2m^6-m^8)l + (-2m^{12}-m^{14})l^2 + (m^{24}+2m^{26})l^4 \\ +(m^{30}-2m^{32}+m^{34})l^5 - m^{38}l^6, \end{aligned}$$

The hyperbolicity equations are

$$\frac{(1-1/d_3)(1-1/b_4)}{(1-c_3)(1-a_4)} = \frac{(1-1/b_6)(1-1/d_7)}{(1-c_6)(1-a_7)} = 1,$$

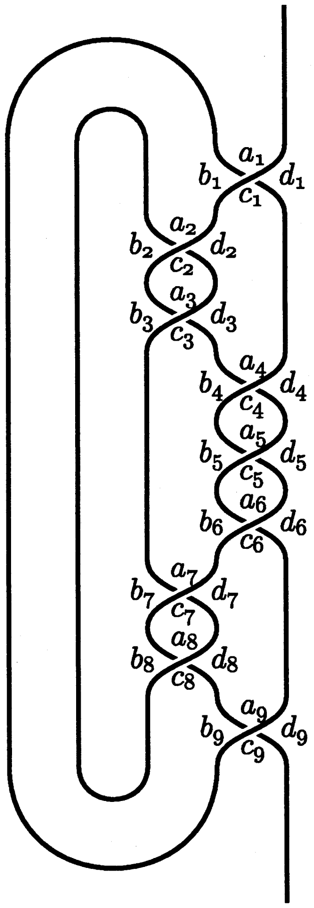

Figure 9:

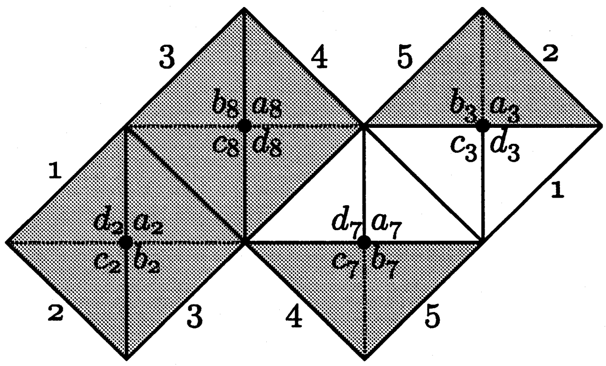

Figure 10:

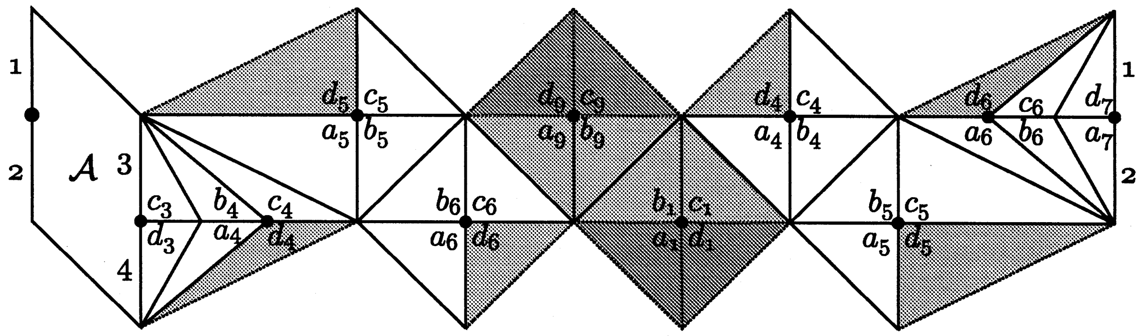

Figure 11:

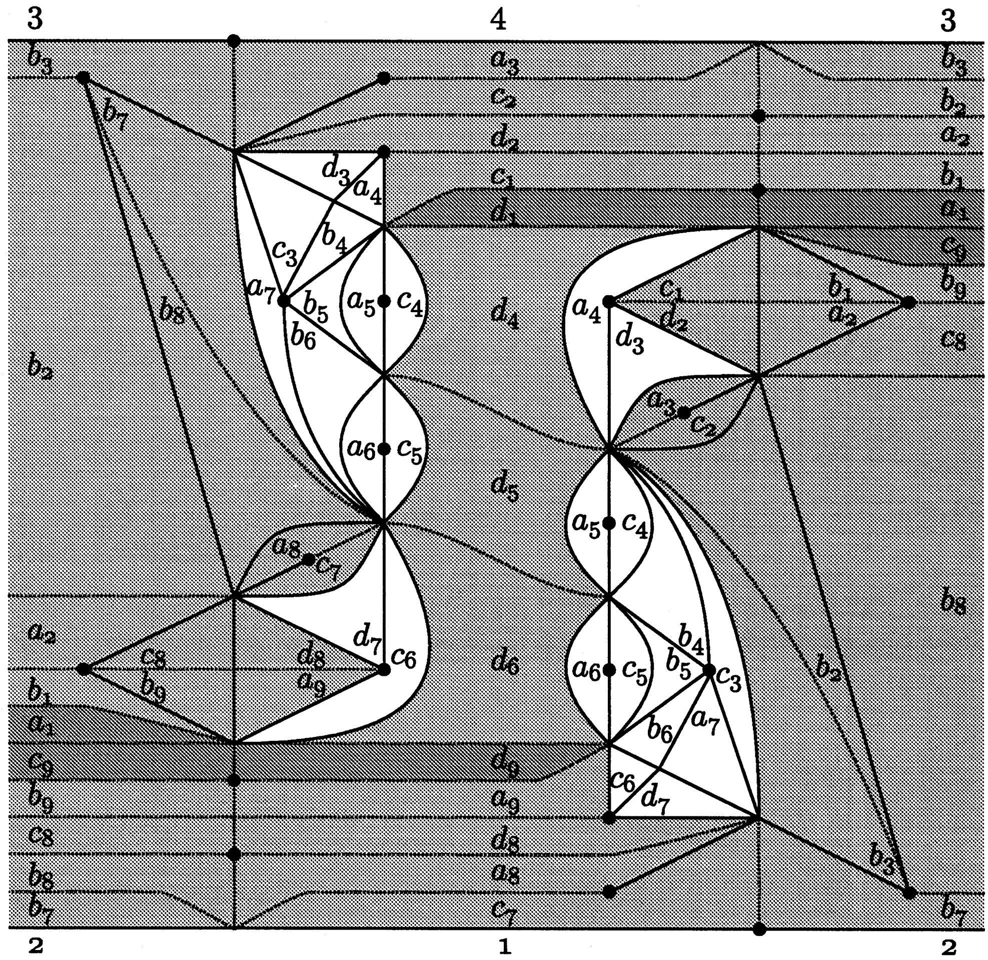

Figure 12:

$$\frac{1-a_7}{1-c_3} = \frac{1-1/d_7}{1-1/d_3} = t,$$

$$\frac{(1-1/b_6)(1-c_5)}{(1-a_6)(1-1/b_5)} = \frac{(1-1/b_5)(1-c_4)}{(1-a_5)(1-1/b_4)} = m,$$

$$s = \frac{c_3 d_3 (1-1/d_3)(1-1/a_7)}{(1-c_3)(1-d_7)} = (c_3 d_3)^2$$

$$l = \frac{c_6 d_7 c_3 d_3}{c_3 b_4 c_4 b_6 c_6 b_5 c_5 \cdot m}$$

The product of the moduli around each edge should be 1, so

$$1 = a_4 b_4 c_4 = a_5 b_5 c_5 = a_6 b_6 c_6 \tag{3}$$

and

$$1 = d_3 a_4 c_6 d_7 = c_3 b_4 b_5 b_6 a_7 = c_4 a_5 = c_5 a_6. \tag{4}$$

From (3) and (4) we can put

$$c_3 = a/x, d_3 = y/a, a_4 = a/m, b_4 = b/a, c_4 = m/b, a_5 = b/m, b_5 = c/b, c_5 = m/c,$$
$$a_6 = c/m, b_6 = d/c, c_6 = m/d, a_7 = x/d, d_7 = d/y.$$

Then we have

$$\frac{(1-a/y)(1-a/b)}{(1-a/x)(1-a/m)} = \frac{(1-c/d)(1-y/d)}{(1-m/d)(1-x/d)} = 1,$$
$$\frac{(1-b/c)(1-m/b)}{(1-b/m)(1-a/b)} = \frac{(1-c/d)(1-m/c)}{(1-c/m)(1-b/c)} = m^2,$$
$$\frac{1-x/d}{1-a/x} = \frac{1-y/d}{1-a/y} = t^2, \quad lm^8 = -bc, \quad s = y/x.$$

By eliminating the variables except l, m, s and t we have following two equations.

$$\begin{aligned} P_1 = &(-1 - m^4 - 3lm^{10} + lm^{12} - l^2m^{12} + 3l^2m^{14} + l^3m^{20} + l^3m^{24}) \\ &+(m^2 - lm^6 + 2lm^8 - 2l^2m^{16} + l^2m^{18} - l^3m^{22})(s^{-1} + s) \\ &+(lm^8 - lm^{10} + l^2m^{14} - l^2m^{16})(s^{-2} + s^2) = 0 \end{aligned}$$

$$\begin{aligned} P_2 = &(l^2m^{12} - 2l^2m^{14} + l^2m^{16})(1 + t^2s^7) \\ &+(2lm^6 - 2lm^8 - 2l^3m^{20} + 2l^3m^{22})(1 + t^2s^5)s \\ &+(1 - lm^8 + lm^{10} + l^2m^{12} - 4l^2m^{14} + l^2m^{16} + l^3m^{18} - l^3m^{20} + l^4m^{28})(1 + t^2s^3)s \\ &+(-m^2 + lm^6 - lm^8 + l^2m^{12} + l^2m^{16} - l^3m^{20} + l^3m^{22} - l^4m^{26})(1 + t^2s)s^3 \\ &+(-lm^8 - 2l^2m^{14} - l^3m^{20})(1 + t^2s^{-1})s^4 = 0. \end{aligned}$$

From these two equations and

$$t = s^n,$$

we obtain the recursive formula of the A-polynomial of K_n.

Theorem 1 *Put*

$$B_n = \begin{cases} -l^2(lm^8)^{3+n}(1-m^2)^n(1+lm^6)^{3+n} & (n > 3), \\ -(lm^8)^{-(2+n)}(1-m^2)^{-(1+n)}(1+lm^6)^{2-n} & (n < 0) \end{cases}$$

and define C_n recursively by

$$\alpha^2 C_n - \alpha\gamma\, C_{n-1} - (2\alpha^2 + 2\alpha\gamma - \beta^2)C_{n-2} - \alpha\gamma C_{n-3} + \alpha^2 C_{n-4} = 0,$$

where

$$\alpha = lm^8(1-m^2)(1+lm^6),\ \beta = m^2 - lm^6 + 2lm^8 - 2l^2m^{16} + l^2m^{18} - l^3m^{22},$$
$$\gamma = -1 - m^4 - 2lm^8 - lm^{10} + lm^{12} - l^2m^{12} + l^2m^{14} + 2l^2m^{16} + l^3m^{20} + l^3m^{24},$$

with the initial conditions

$$C_0 = -\frac{lm^8\{A_{K_0}(l,m)\}^2}{(1+lm^6)^2}, \qquad C_1 = \frac{m^4(1-lm^8)\{A_{K_1}(lm^{-4},m)\}^2}{(1-m^2)(1+lm^6)},$$
$$C_2 = -\frac{\{A_{K_2}(lm^{-8},m)\}^2}{l(1-m^2)^2}, \qquad C_3 = \frac{A_{K_3}(lm^{-12},m)}{l^2m^4(1-m^2)^3}.$$

Then, $A_{K_n}(lm^{-4n},m)$ is a factor of $B_nC_n \in \mathbf{Z}[l,m]$ for $n > 3$ and $n < 0$.

Remark *In fact, when $n \equiv 1 \bmod 3$, B_nC_n contains the factor $1 - lm^8$ but it is not a factor of the A-polynomial of K_n.*