Link invariant as critical values of the dilogarithm functions

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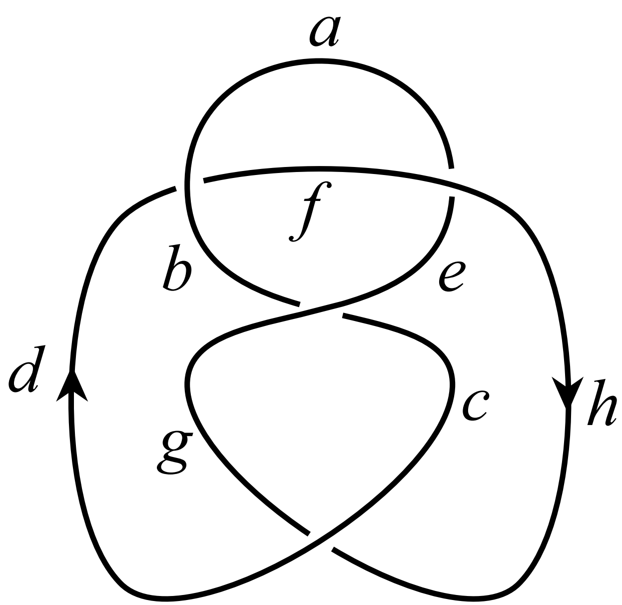
Abstract.

We get a link invariant in following steps.



$\longrightarrow$ The function $F_D : \mathbb{C}^m \to \mathbb{C}$.

The diagram D of a link.

The set of critical values of F_D is invariant under the Reidemeister moves. This set for a figure eight knot 4_1 includes the volume of $S^3 - 4_1$.

The idea of constructing F_D.

Conjecture. Let L be a hyperbolic link and $J_N(L)$ be the colored Jones polynomial obtained from the N-dim. irr. rep. of $U_q(sl_2)$.

$$J_N(L) \sim \exp \frac{N}{2\pi}\left(\mathrm{Vol} + \mathrm{CS}\sqrt{-1}\right) \qquad (N \to \infty)$$

Optimistic calculation

$\boxed{\text{The colored Jones polynomial}}$

$\downarrow$ Approximation and suddle point method.

$\boxed{\text{The complex valued function}}$

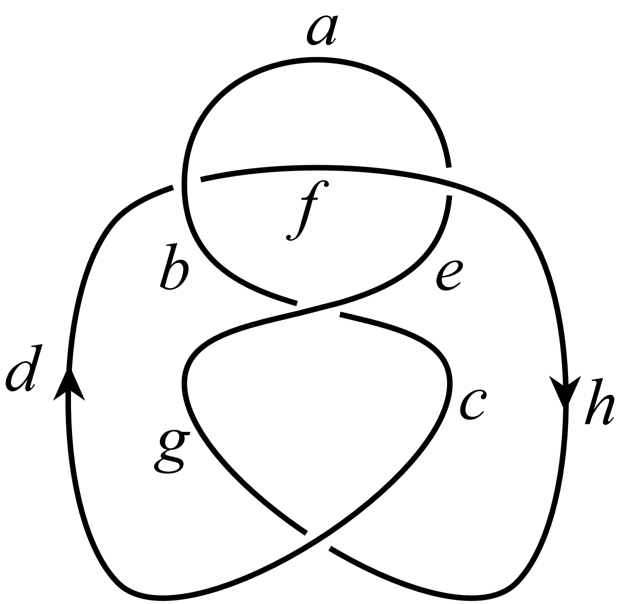
$\downarrow$ Finding the critical values.

$\boxed{\text{The volume and the Chern-Simons invariant}}$

Definition.

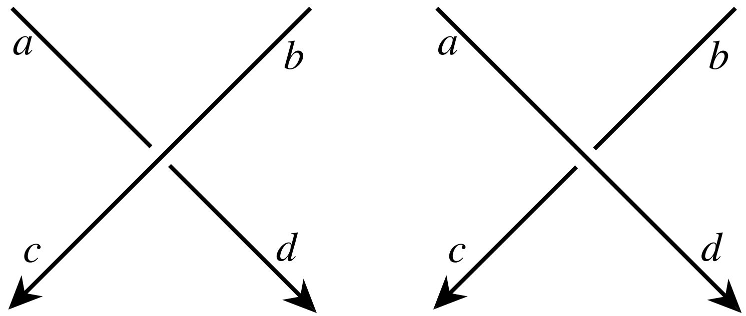
Let L be a link and D be it's diagram.

First we associate each edge in D with parameters.



The diagram D of 4_1.

Here we associate parameters, subject to $ab = cd$ in each crossing as below.



$(X_+)^{a,b}_{c,d}$ $(X_-)^{a,b}_{c,d}$

In the above diagram D we can put $f = \dfrac{ad}{b}$, $g = \dfrac{ce}{b}$ and $h = \dfrac{de}{b}$.

Next we associate a positive crossing $(X_+)^{a,b}_{c,d}$ and a negative crossing $(X_-)^{a,b}_{c,d}$ with functions defined as follows.

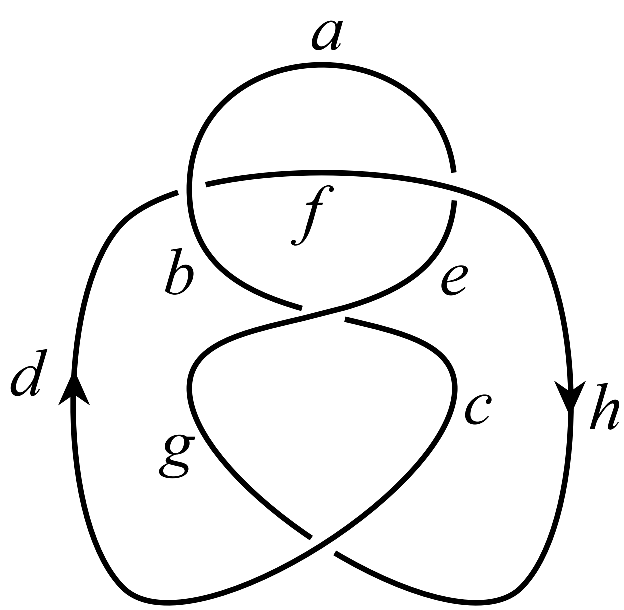
$$(X_+)^{a,b}_{c,d} = \mathrm{Li}_2(a) + \mathrm{Li}_2(1/b) + \mathrm{Li}_2(c) + \mathrm{Li}_2(1/d) - \mathrm{Li}_2(a/d) + \frac{(\log(-ab))^2}{2},$$

$$(X_-)^{a,b}_{c,d} = -\mathrm{Li}_2(a) - \mathrm{Li}_2(1/b) - \mathrm{Li}_2(c) - \mathrm{Li}_2(1/d) + \mathrm{Li}_2(a/d) - \frac{(\log(-ab))^2}{2},$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt.$$

We define the function F_D as the sum of above functions for all crossings of D



The diagram D of 4_1. $f = \dfrac{ad}{b}$, $g = \dfrac{ce}{b}$ and $h = \dfrac{de}{b}$.

For example if D is the diagram of 4_1, F_D is

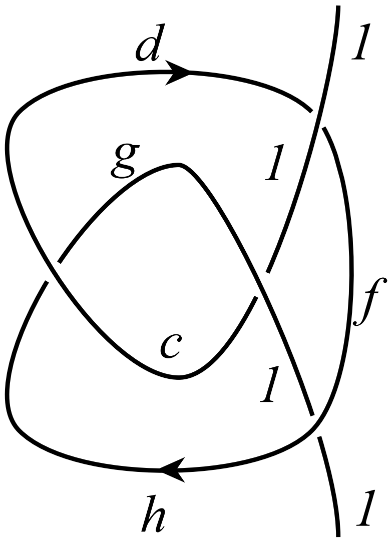
$$F_D(a,b,c,d,e) = (X_+)^{d,a}_{b,f} + (X_+)^{f,e}_{a,h} + (X_-)^{g,b}_{c,e} + (X_-)^{c,h}_{g,d}$$

Theorem.

The set of critical values of F_D is invariant under the Reidemeister moves. So this set is a link invariant and denoted by $\mathcal{V}(L)$.

Remark 1.

Let T be a (1,1) tangle presentation of L obtained by cutting a component of L.



$$\longrightarrow \quad F_T(c = g, d = f = h)$$
$$= (X_+)^{d,1}_{1,f} + (X_-)^{1,f}_{h,1} + (X_+)^{g,1}_{c,1} + (X_-)^{c,h}_{g,d}$$

F_T coincides with the function obtained from the colored Jones polynomial by the optimistic calculation.

Remark 2.

There is a nine-term three-variable functional equation,

$$\mathrm{Li}_2\left(\frac{vw}{xy}\right) = \mathrm{Li}_2\left(\frac{v}{x}\right) + \mathrm{Li}_2\left(\frac{w}{y}\right) + \mathrm{Li}_2\left(\frac{v}{y}\right) + \mathrm{Li}_2\left(\frac{w}{x}\right)$$
$$+\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2(v) - \mathrm{Li}_2(w) + \frac{1}{2}(\log(-x/y))^2,$$

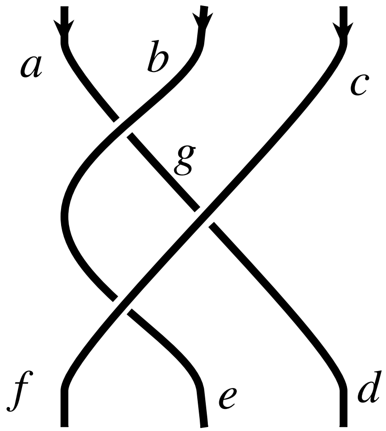
subject to the constraint $(1-v)(1-w) = (1-x)(1-y)$.

Put x, y, z and w as functions of a, b, c and d, and we have

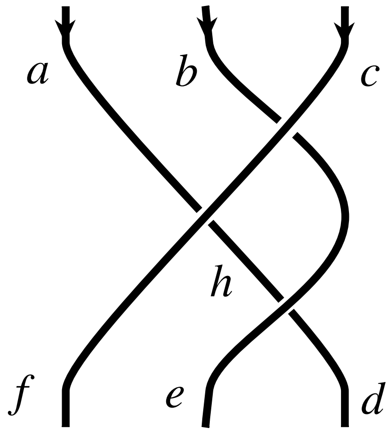
$$(X_+)^{ab}_{cd} = -\mathrm{Li}_2(x) - \mathrm{Li}_2(y) + \mathrm{Li}_2(v) + \mathrm{Li}_2(w).$$

Invariance under the Reidemeister move III.

Let D and D' be two diagrams of a link and D' is obtained by the Reidemeister move III.



D



D'

Here $f = \dfrac{abc}{de}$.

From definition we can put $F_D = F_1(a, b, c, d, e, g) + G$ and $F_{D'} = F_2(a, b, c, d, e, h) + G$ where G is the function of parameters except g and h.

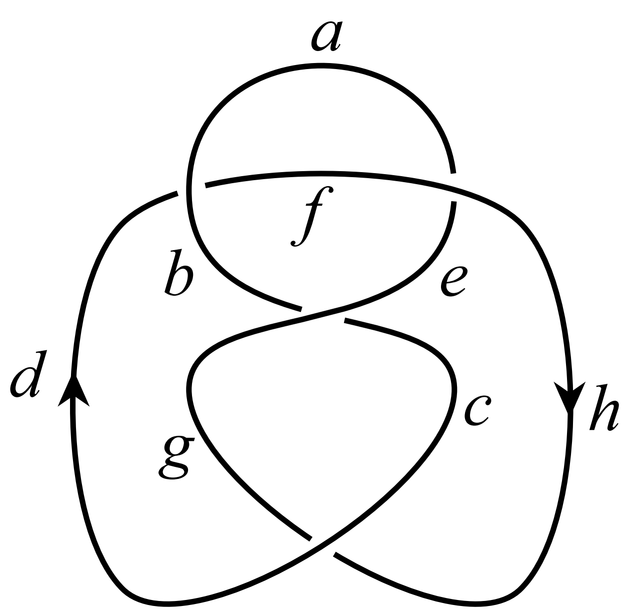
$$\frac{\partial F_D}{\partial g} = \frac{\partial F_1}{\partial g} = \frac{1}{g}\log\left(\frac{e(ab-g)^2(g-d)}{g(a-g)(ab-eg)}\right),$$
$$\frac{\partial F_{D'}}{\partial h} = \frac{\partial F_2}{\partial h} = \frac{1}{h}\log\left(\frac{ae(d-h)(de-bh)}{(a-h)(de-h)^2}\right).$$

Two equations $\exp\left(g\dfrac{\partial F_1}{\partial g}\right) = 1$ and $\exp\left(h\dfrac{\partial F_2}{\partial h}\right) = 1$ are reduced quadratic equations and put g_1, g_2, h_1 and h_2 be the solutions of these equations. By simple calculation we have

$$\begin{aligned}\left.\frac{\partial(F_D - F_{D'})}{\partial a}\right|_{g=g_1,h=h_1} &= \left.\frac{\partial(F_D - F_{D'})}{\partial b}\right|_{g=g_1,h=h_1} = \left.\frac{\partial(F_D - F_{D'})}{\partial c}\right|_{g=g_1,h=h_1} \\ &= \left.\frac{\partial(F_D - F_{D'})}{\partial d}\right|_{g=g_1,h=h_1} = \left.\frac{\partial(F_D - F_{D'})}{\partial e}\right|_{g=g_1,h=h_1} = 0\end{aligned}$$

This follows $F_D\Big|_{g=g_1} = F_{D'}\Big|_{h=h_1}$.

Figure eight knot.



$$\longrightarrow \quad F_D(a, b, c, d, e)$$

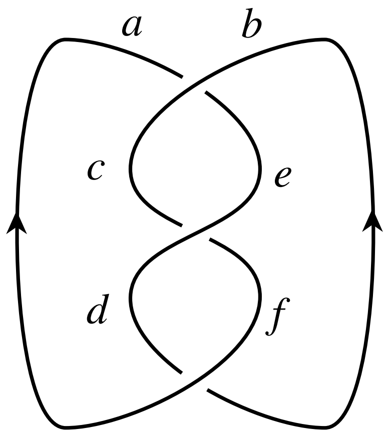
We can solve

$$\exp\left(a\frac{\partial F_D}{\partial a}\right) = \exp\left(b\frac{\partial F_D}{\partial b}\right) = \cdots = \exp\left(e\frac{\partial F_D}{\partial e}\right) = 1,$$

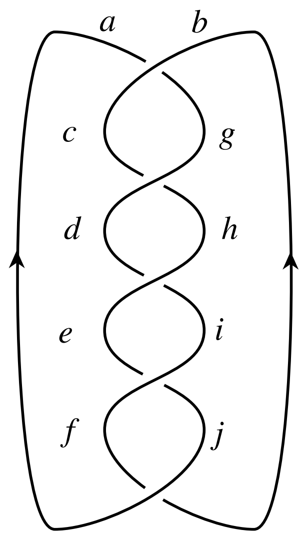
with respect to c, d and e as functions of a and b. Numerically we have

$$\mathrm{CS}(S^3 - 4_1) + \mathrm{Vol}(S^3 - 4_1)\sqrt{-1} \in \mathcal{V}(4_1).$$

Torus knots.



$T(2,3)$.



$T(2,5)$.

Numerically we have

$$\frac{\pi^2}{6} \in \mathcal{V}(T(2,3)), \ \frac{\pi^2}{10} \in \mathcal{V}(T(2,5)), \ \frac{\pi^2}{14} \in \mathcal{V}(T(2,7)).$$

Problems.

- For a hyperbolic link L,

$$CS(S^3 - L) + \mathrm{Vol}(S^3 - L)\sqrt{-1} \in \mathcal{V}(L).$$

- For a torus knot $T(s, t)$,

$$\frac{\pi^2}{st} \in \mathcal{V}(T(s, t)).$$