Volumes of hyperbolic tetrahedra and quantum 6j-symbols

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Hyperbolic Volumes

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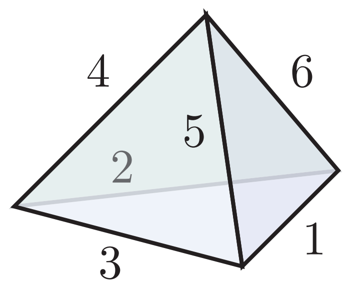
1. Volume of tetrahedron

1.1. Volume from angles

For orthoschemes, a closed formula is known from long time before.

For generic tetrahedra, the first result is given by

Y.Cho and H.Kim
Discrete Comput. Geom.
22 (1999), 347
(decomposition by ideal tetrahedra)



This result is modified to more symmetric form by
J.M. and M. Yano
to appear in
Commun. in Analysis and Geometry
(from quantum 6j-symbol)

Notations: $\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-x)}{x},$

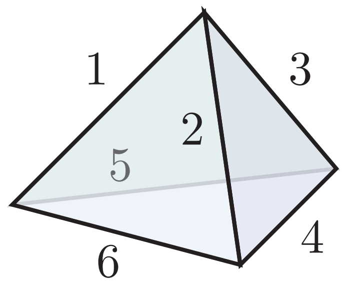
$$g(z) = \frac{\pi\sqrt{-1}}{2}\log z - \frac{(\log z)^2}{4} - \mathrm{Li}_2(z),$$

$$w(\mathbf{z}; x) = \pi\sqrt{-1}\log x + g(x)$$
$$-g\left(\frac{x}{z_1 z_2 z_3}\right) - g\left(\frac{x}{z_1 z_5 z_6}\right) - g\left(\frac{x}{z_2 z_4 z_6}\right) - g\left(\frac{x}{z_3 z_4 z_5}\right)$$
$$-g\left(\frac{z_1 z_2 z_4 z_5}{x}\right) - g\left(\frac{z_1 z_3 z_4 z_6}{x}\right) - g\left(\frac{z_2 z_3 z_5 z_6}{x}\right),$$

$$\frac{\partial}{\partial x} w(\mathbf{z}; x_i) = 0, \quad (i = 1, 2,\ x_i \neq 0)$$

$$U(\mathbf{z}) = \frac{w(\mathbf{z}; x_1) - w(\mathbf{z}; x_2)}{2},$$

Let $z_p = -\exp\sqrt{-1}\,\theta_p$, then



Theorem. (J.M.-M.Y.)

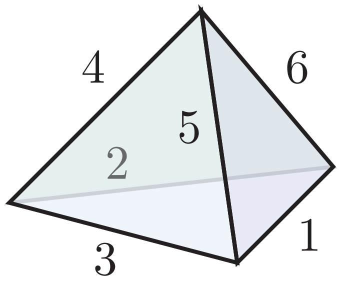
$$2\,\mathrm{Vol}(T) = \sigma\, U(\mathbf{z}).$$

where $\sigma = 1$ (elliptic), $\sqrt{-1}$ (hyperbolic).

1.2. Volume from lengths

Hyperbolic case:

For a hyperbolic tetrahedron, let



$$z_p = \exp l_p.$$

Theorem. (J.M.-A.Ushijima)

$$2\,\mathrm{Vol}(T) = \pm\sqrt{-1}\left(U(\mathbf{z}) - \sum_{p=1}^{6} l_p \frac{\partial(U(\mathbf{z}))}{\partial l_p}\right),$$

$$\mathbf{z} = (z_1, \cdots, z_6),$$

$$\theta_p = \pm\sqrt{-1}\frac{\partial}{\partial l_p}U(\mathbf{z}) \ : \text{ dihedral angle},$$

$$U(\mathbf{z}) \in \sqrt{-1}\,\mathbf{R}.$$

C.f. Santalo's relation (explain later)

Elliptic case:

For a tetrahedron in a 3-sphere S^3, let

$$z_p = \exp \sqrt{-1}\, l_p.$$

Theorem.

$$2\,\mathrm{Vol}(T) \;=\; \pm \left(U(\mathbf{z}) + \sum_{p=1}^{6} l_p \frac{\partial(U(\mathbf{z}))}{\partial l_p} \right),$$

$$\theta_p = \frac{\partial}{\partial l_p} U(\mathbf{z}) \;:\; \text{dihedral angle}$$

$$U(\mathbf{z}) \in \mathbf{R}$$

C.f. Milnor's relation:

$$\mathrm{Vol}(T) + \mathrm{Vol}(T^*) + \frac{1}{2} \sum l_p \theta_p = \pi^2.$$

where T^* is the dual tetrahedron in S^3 whose edge lengths are $\pi - \theta_p$ and dihedral angles are $\pi - l_p$.

Construction of P^*.

P^* is a set of all points of S^3 whose distances from P are at least $\pi/2$.

Milnor's relation implies the above theorem for elliptic case.

Stantalo's relation.

Hyperbolic version of Milnor's relation.
T^* is a vertual (imaginary) tetrahedron.

Santalo's relation should imply the above theorem for hyperbolic case if we have a good interpretation of T^*.

A relation between dihedral angles and edge lengths of T is Known, and the formula is proved by using it.
(His talk)

2. Background of the formula

2.1. Volume conjecture

For knots

R.Kashaev :

certain knot invariants $\longrightarrow$ hyperbolic volume

$\|$

H.Murakami-J.M. :

colored Jones Poly. $J_N \longrightarrow$ simplicial volume

H.M.-J.M.-M.Okamoto-T.Takata-Y.Yokota :

$J_N \longrightarrow$ hyperbolic volume $+\sqrt{-1}$ CS

For 3-manifolds

Witten-Reshetikhin-Turaev inv. of 3-mfd.

$\xrightarrow[\text{H.Murakami}]{\qquad\qquad}$ hyperbolic volume $+\sqrt{-1}$ CS

$\downarrow$ absolute value^2 $\quad\nearrow$ **?**

Turaev-Viro invariant

$\nwarrow$

simplicial decompositon

2.2. Quantum $6j$-symbol

Notations: Fix $N \geq 3$ (integer)

$$I = \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{N-3}{2}, \frac{N-2}{2}\}$$

$$q_0 : \text{a } 2N\text{-th root of unity}, \; q = {q_0}^2$$

$$\text{For } n \geq 1, \quad [n] = \frac{{q_0}^n - {q_0}^{-n}}{q_0 - {q_0}^{-1}} \quad \text{(q-integer)}$$

$$[n]! = [n][n-1]\cdots[2][1] \qquad \text{(q-factorial)}$$

$$(i, j, k) \in I^3 \text{ is } \textbf{addmissible} \Leftrightarrow \begin{cases} i \leq j + k, \\ j \leq k + i, \\ k \leq i + j, \\ i + j + k \leq N - 2 \end{cases}$$

$$\Delta(i, j, k) = \sqrt{\frac{[i+j-k]![j+k-i]![k+i-j]!}{[i+j+k+1]!}}$$

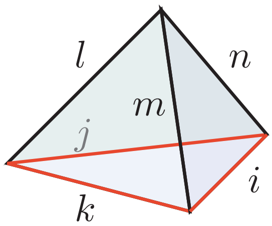
Rakah-Wigner symbol

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW} =$$

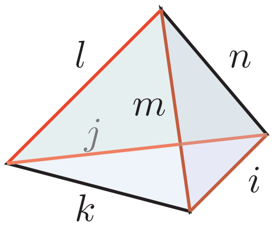
$$\Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m) \times$$

$$\sum_r \frac{(-1)^r \, [z+1]!}{[r-a_1]! \cdots [r-a_4]! [b_1 - r]! \cdots [b_3 - r]!}$$

$$a_1 = i + j + k$$
$$a_2 = i + m + n$$
$$a_3 = j + l + n$$
$$a_4 = k + l + m$$



$$b_1 = i + j + l + m$$
$$b_2 = i + k + l + n$$
$$b_3 = j + k + m + n$$



$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW}$$

$\longrightarrow$ **volume of** T

2.3 Turaev-Viro invariant

Notations:

M : 3-manifold,

$\mathcal{T}$: tetrahedral decompsition of M

a : number of vertices

b : number of edges

$\varphi : \{\text{edges of } M\} \longrightarrow I$ **(coloring)**

$$u_i = \left(\sqrt{-1}\right)^{2i} [2i+1]^{\frac{1}{2}}, \quad u = \frac{2r}{\left|q_0 - q_0{}^{-1}\right|^2}$$

$$|M|_\varphi = u \prod_{E\,:\,\text{edge}} u_{\varphi(E)} \prod_{T\,:\,\text{tetrahedron}} |\varphi(T)|$$

$$|M| = \sum_{\varphi\,:\,\text{coloring}} |M|_\varphi$$

Theorem (Turaev-Viro)

$|M|$ is a topologial invariant of M.

2.4 Volume conjecture for 3-mfd.

$\tau_N(M)$: Witten-Reshetikhin-Turaev invariant

Conjecture (H.Murakami, checked also by K.Ohnuki)

$$\underset{N\to\infty}{\text{o-lim}} \frac{2\pi \log \tau_N(M)}{N} = \text{Vol}(M) + \sqrt{-1}\,\text{CS}(M)$$

Known relation. $\left|\tau_N(M)\right|^2 = |M|$

Modified conjecture.

$$\underset{r\to\infty}{\text{o-lim}} \frac{\pi \log |M|}{N} = \text{Vol}(M)$$

Yokota's theory.
For a hyperbolic knot complement,

o-lim $\longleftrightarrow$ geometrization condition

saddle point hyperbolicity equation

What about a closed 3-manifold ?

2.5 Optimistic limit

Step 1. *Operation R:* Replace

$$[n]! \longrightarrow \exp \frac{N}{2\pi\sqrt{-1}}\, g(z_n)$$

$$g(z_n) \underset{\text{def}}{=} \frac{\pi\sqrt{-1}}{2}\log z_n - \frac{(\log z_n)^2}{4} - \mathrm{Li}_2(z_n)$$

$$\left(z_n = q^n = \exp \frac{2\pi n\sqrt{-1}}{N}, \quad \mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-x)}{x} \right)$$

$$n = \frac{N}{2\pi\sqrt{-1}} \log z_n,$$

$$\log([n]!) = -n\log\left(-q_0 + {q_0}^{-1}\right) + \sum_{k=1}^{n} \log(-{q_0}^k + {q_0}^{-k})$$

$$= -n\log\left(-q_0 + {q_0}^{-1}\right) - \frac{n(n+1)}{2}\log q_0 + \sum_{k=1}^{n} \log(1-q^k)$$

$$\sim \frac{N}{4}\log z_n - \frac{N}{8\pi\sqrt{-1}}(\log z_n)^2 + \int_0^n \log\left(1 - e^{2\pi t\sqrt{-1}/N}\right) dt$$

$$\sim \frac{N}{2\pi\sqrt{-1}}\left(\frac{\pi\sqrt{-1}}{2}\log z_n - \frac{(\log z_n)^2}{4} + \int_1^{z_n} \frac{\log(1-x)}{x}\, dx \right)$$

$$= \frac{N}{2\pi\sqrt{-1}}\left(\frac{\pi\sqrt{-1}}{2}\log z_n - \frac{(\log z_n)^2}{4} - \mathrm{Li}_2(z_n) \right)$$

Step 2. *Operation S:*

Apply **Saddle point method:**

$f(z_1, \cdots, z_n)$: holomorphic function

$$\left| \int_{z_1,\cdots,z_n} \exp \sqrt{-1} N f(z_1, \cdots, z_n) \right| \underset{N\to\infty}{\sim} \exp N \left| \sqrt{-1} f(z_1^0, \cdots, z_n^0) \right|,$$

where $z_1^0, \cdots, z_n^0$ satisfy

$$\frac{\partial}{\partial z_i} f(z_1^0, \cdots, z_n^0) = 0 \qquad (i = 1, \cdots, n)$$

+ condition about the integral path
(e.g. steepest decent)

In the volume conjedture for 3-manifolds, such condition is **ignored**. $\longrightarrow$ **Oprimistic limit**

Remark.
There may be several oprimistic limits.

2.6. Formula by length

Apply to 6j-symbol $\longrightarrow$ formula

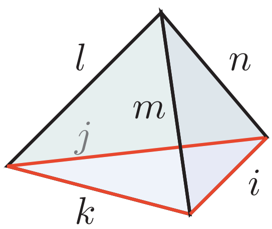
Optimistic limit of $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$

($z_i = q^i,\ z_j = q^j,\ \cdots,\ z_n = q^n$ are fixed.)

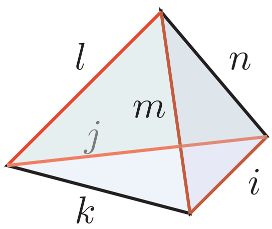
$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW},$$

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW} =$$
$$\Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m) \times \sum_r \frac{(-1)^r\,[z+1]!}{[r-a_1]!\cdots[r-a_4]![b_1-r]!\cdots[b_3-r]!}.$$

$$\begin{aligned} a_1 &= i+j+k \\ a_2 &= i+m+n \\ a_3 &= j+l+n \\ a_4 &= k+l+m \end{aligned}$$



$$\begin{aligned} b_1 &= i+j+l+m \\ b_2 &= i+k+l+n \\ b_3 &= j+k+m+n \end{aligned}$$



3. Geometric structure

3.1. o-lim of Turaev-Viro inv.

Recall **notations:**

M : 3-manifold,

$\mathcal{T}$: tetrahedral decompsition of M

a : number of vertices

b : number of edges

$\varphi : \{\text{edges of } M\} \longrightarrow I$ (**coloring**)

$$u_i = \left(\sqrt{-1}\right)^{2i} [2i+1]^{\frac{1}{2}}, \quad u = \frac{2r}{\left|q_0 - q_0{}^{-1}\right|^2}$$

$$|M|_\varphi = u \prod_{E\ :\ \text{edge}} u_{\varphi(E)} \prod_{T\ :\ \text{tetrahedron}} |\varphi(T)|$$

$$|M| = \sum_{\varphi\ :\ \text{coloring}} |M|_\varphi \quad \text{(Turaev-Viro Invariant)}$$

Modified conjecture.

$$\underset{r\to\infty}{\text{o-lim}} \frac{\pi \log |M|}{N} = \text{Vol}(M)$$

Operation R, S:

$z(E)$: parameter corresponding to E

$$|M| \underset{\text{Operation R}}{\longrightarrow} \underbrace{\exp\left(\frac{\sqrt{-1}N}{2\pi}\times \int_{\substack{z_E \\ E:\text{edge}}} - \sum_{E:\text{edge}} 2\pi\sqrt{-1}\log z_E + \sum_{T:\text{tetrahedron}} U(T)\right)}_{R}.$$

$$\text{Operation S} \quad \downarrow \quad \text{value at saddle point } z_E^0$$

z_E^0 : solution of $\dfrac{\partial R}{\partial z_E} = 0$ (for all edges E),

$$\text{i.e.} \quad \frac{2\pi\sqrt{-1}}{z_E} = \sum \frac{\partial U}{\partial z_E} = \sum_{T\ni E} \frac{\partial U(T)}{\partial z_E} = \sum_{T\ni E} \sqrt{-1}\,\frac{\theta_E(T)}{z_E}.$$

Suppose that $\log z_E^0$ is real and is the length of E. Then

$$R_0 = \sum_E 2\pi\sqrt{-1}\log z_E^0 + \sum_{T:\text{tetrahedron}} U_i(T)\Big|_{z_E = z_E^0}$$

$$= \sum_{T:\text{tetrahedron}} \widetilde{U}_i(T)\Big|_{z_E = z_E^0}$$

$$\underset{\text{conjecture}}{=} \boxed{\sum_{T:\text{tetrahedron}} 2\sqrt{-1}\,\mathrm{Vol}_0(T)}\,.$$

If $l_E^0 (= \log z_E^0)$ is a **positive real** number for each edge E and satisfy the trigonometric inequarity, these parameters determin a hyperbolic structhre of M.

4.2. Casson's algorithm

Schläfli's formula T : tetrahedron,

$$l_E = \frac{\varepsilon}{2} \frac{\partial \mathrm{Vol}(P)}{\partial \theta_E} \quad \left(\varepsilon = \begin{cases} 1 & \text{elliptic (spherical)}, \\ -1 & \text{hyperbolic}. \end{cases} \right)$$

M : hyperbolic 3-manifold,
$\mathcal{T}$: tetrahedral decompsition of M,
$\varphi : \{\text{edges}\} \to \mathbf{R}$ (length).

$$P_\varphi(M) = \sum_{T \,:\, \text{tetrahedron}} \mathrm{Vol}(T_\varphi) + \frac{1}{2} \sum_{E \,:\, \text{edge}} \theta_E \varphi_E$$

φ^0 : solution of $\dfrac{\partial P_\varphi}{\partial \varphi_E} = \pi.$

Argorithm. (Casson)
φ^0 often determines the hyperbolic structure of M.

Remark.

$P_\varphi(M)$ corresponds to the previous $\sum_T U(T)$.

$$|M| \underset{\text{Operation R}}{\longrightarrow} \underbrace{\exp\left(\frac{\sqrt{-1}N}{2\pi} \times \int_{\substack{z_E \\ E:\text{edge}}} - \sum_{E:\text{edge}} 2\pi\sqrt{-1}\log z_E + \sum_{T:\text{tetrahedron}} U(T)\right)}_{R}.$$

$$\begin{aligned}\frac{\partial P_\varphi}{\partial \varphi_E} &= \sum_k \frac{\partial\,\text{Vol}(T^k)}{\partial \theta_E^k}\frac{\partial \theta_E^k}{\partial \varphi_E} + \sum_k \frac{1}{2}\frac{\partial \theta_E^k}{\partial \varphi_E}\varphi_E + \sum_k \frac{1}{2}\theta_E^k \\ &= \sum_k \frac{1}{2}\theta_E^k \left(= \pi \text{ at } \varphi^0\right)\end{aligned}$$

Saddle point of R corresponds to φ^0, a solution of $\dfrac{\partial P_\varphi}{\partial \varphi_E} = \pi$.

Conclusion

1. The quantum $6j$-symbol determines the volume of tetrahedron.

2. The Turaev-Viro invariant may determine the hyperbolic structure of M as well as Casson's argorithm.

Ploblem.

What about 3-manifold with **general** geometric structure?

An idea:

Modify Casson's algorithm in terms of dihedral angles instead of edge lengths, then the saddle point may give the geometric structure for the general case.