Scissors Congruence: Volume Symmetries

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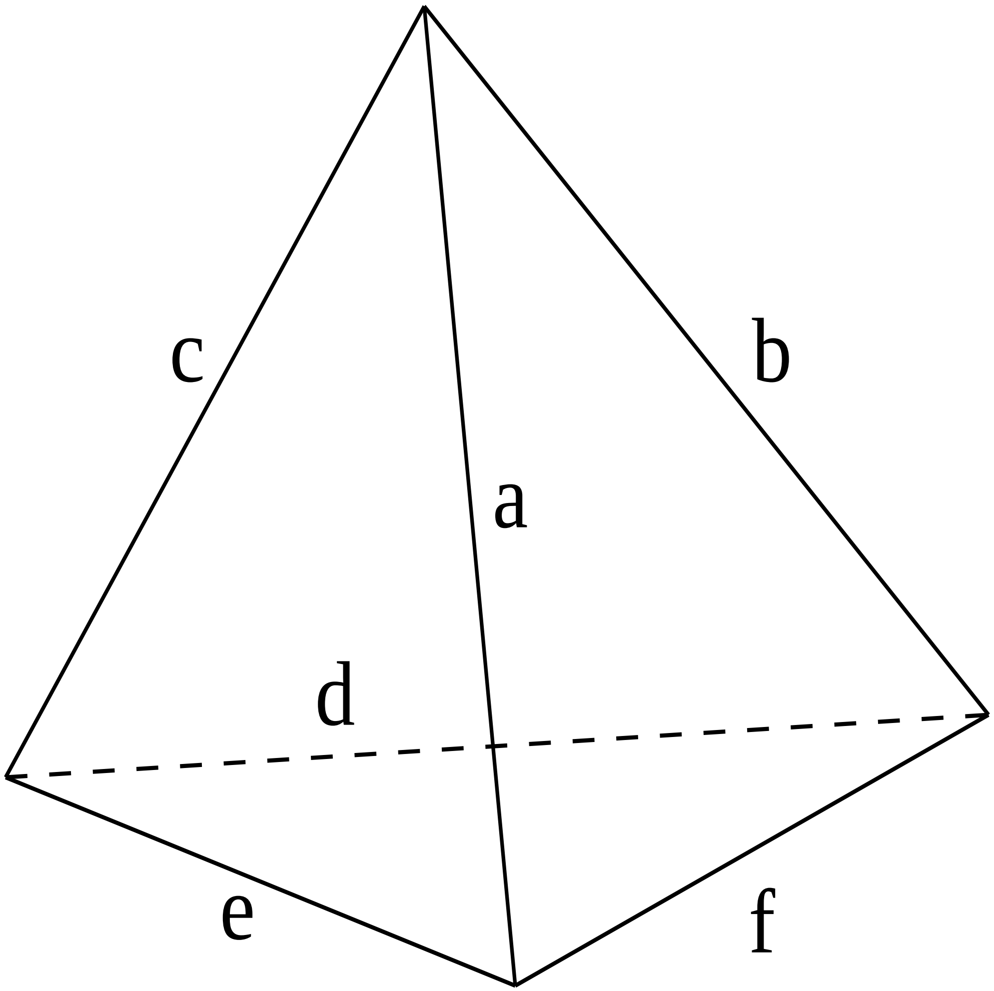
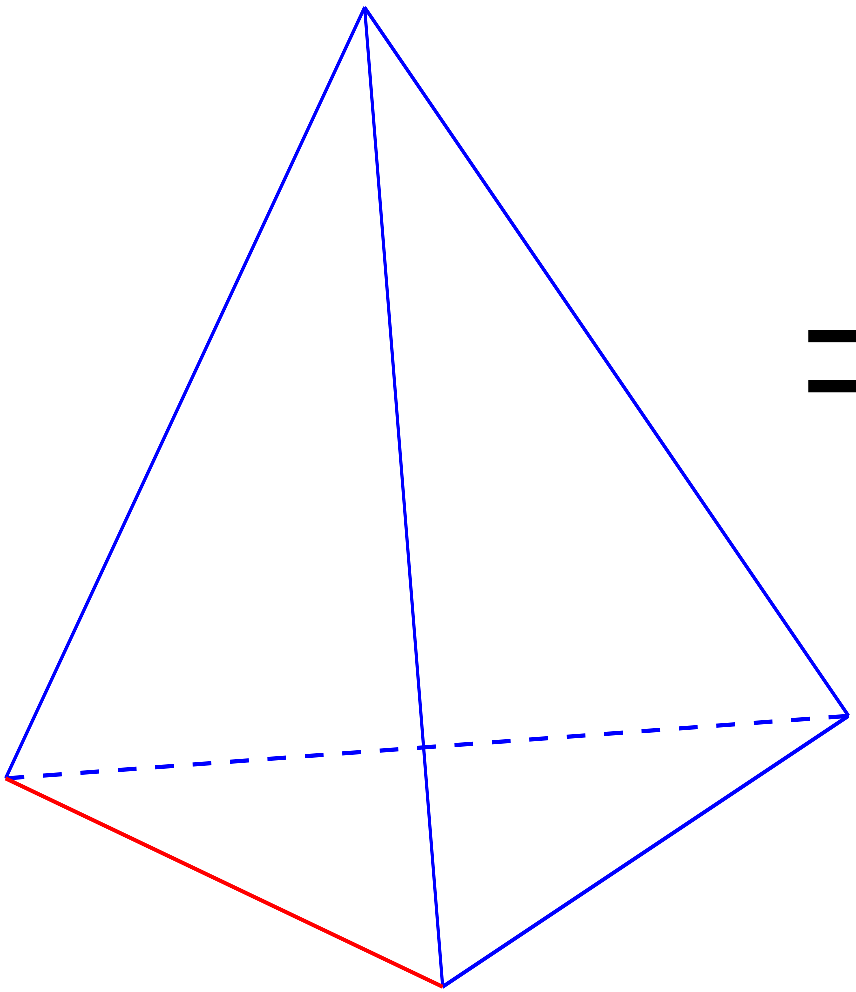
Notation

The labeled angles are the *circulants* of the dihedral angles, where $e^{I\theta}$ is the circulant of the dihedral angle θ.



Notation

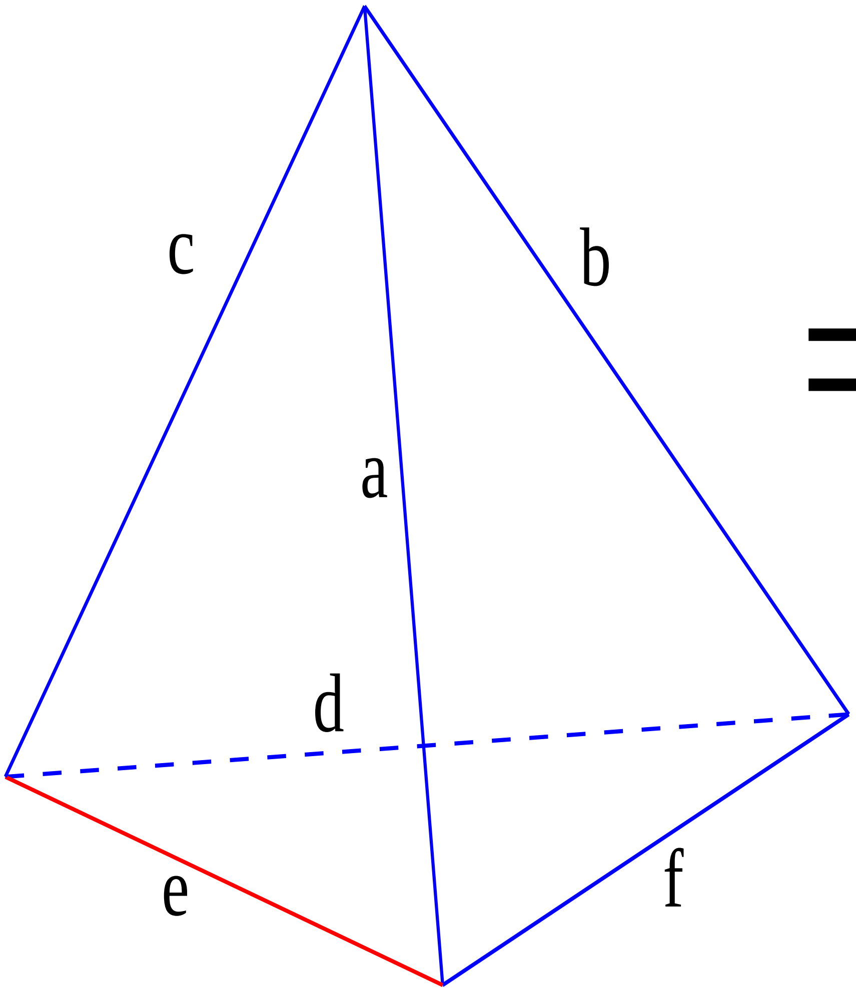
We write monomial expression in these circulants using this template. For example we have this expression.



$= e$

Notation

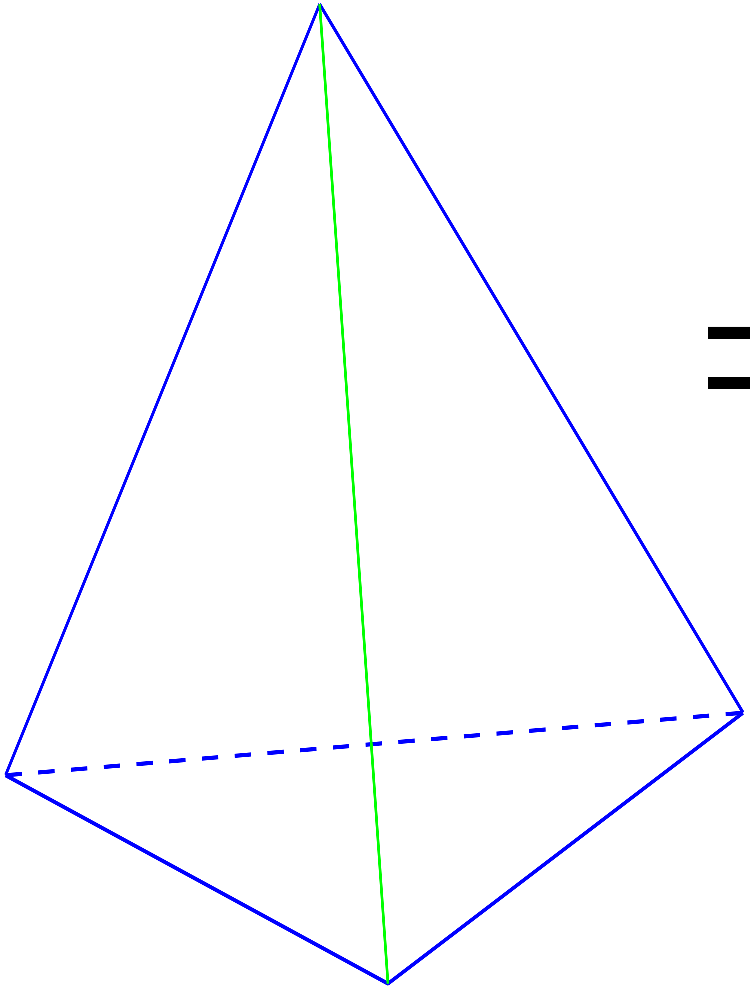
The interpretation of which requires us to recall the circulants from the template.



$= e$

Notation

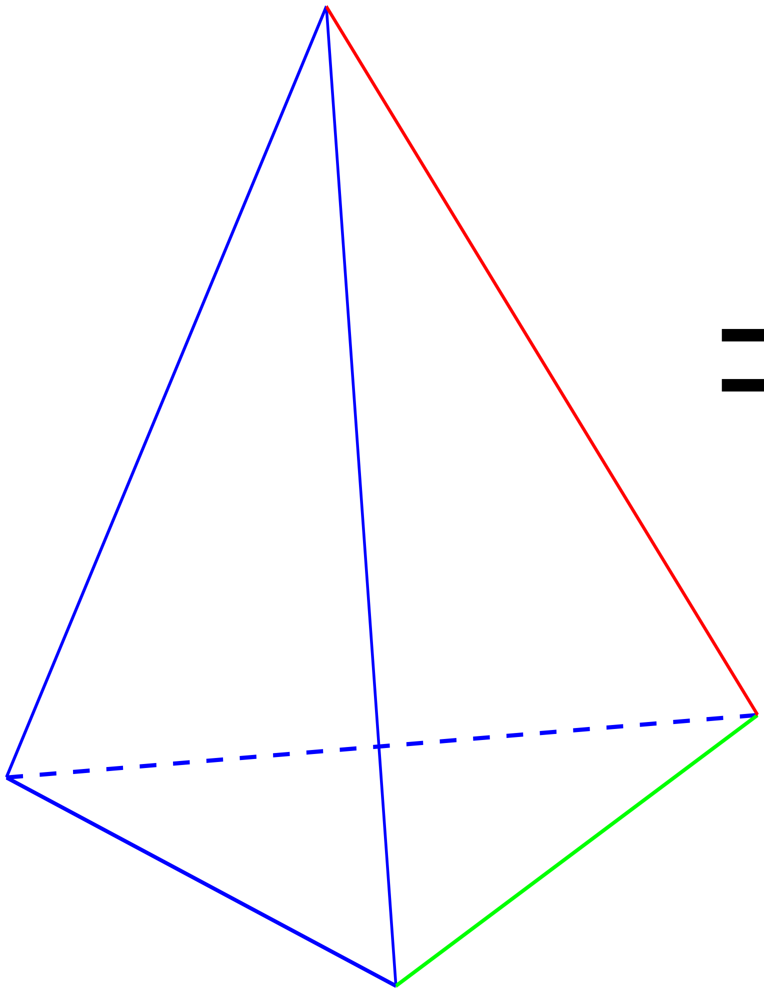
Use of a green edge means take the reciprocal.



$=1/a$

Notation

Coloring multiple edges corresponds to taking the product.

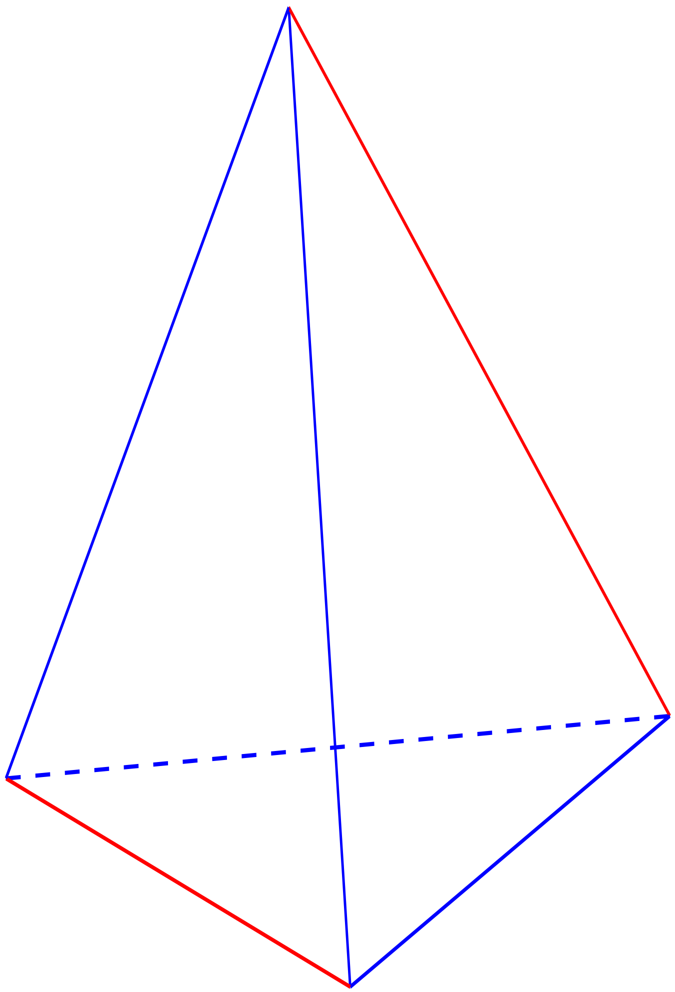


$= b/f$

Notation

All sums will mean over all orientation preserving tetrahedral symmetries. For example:

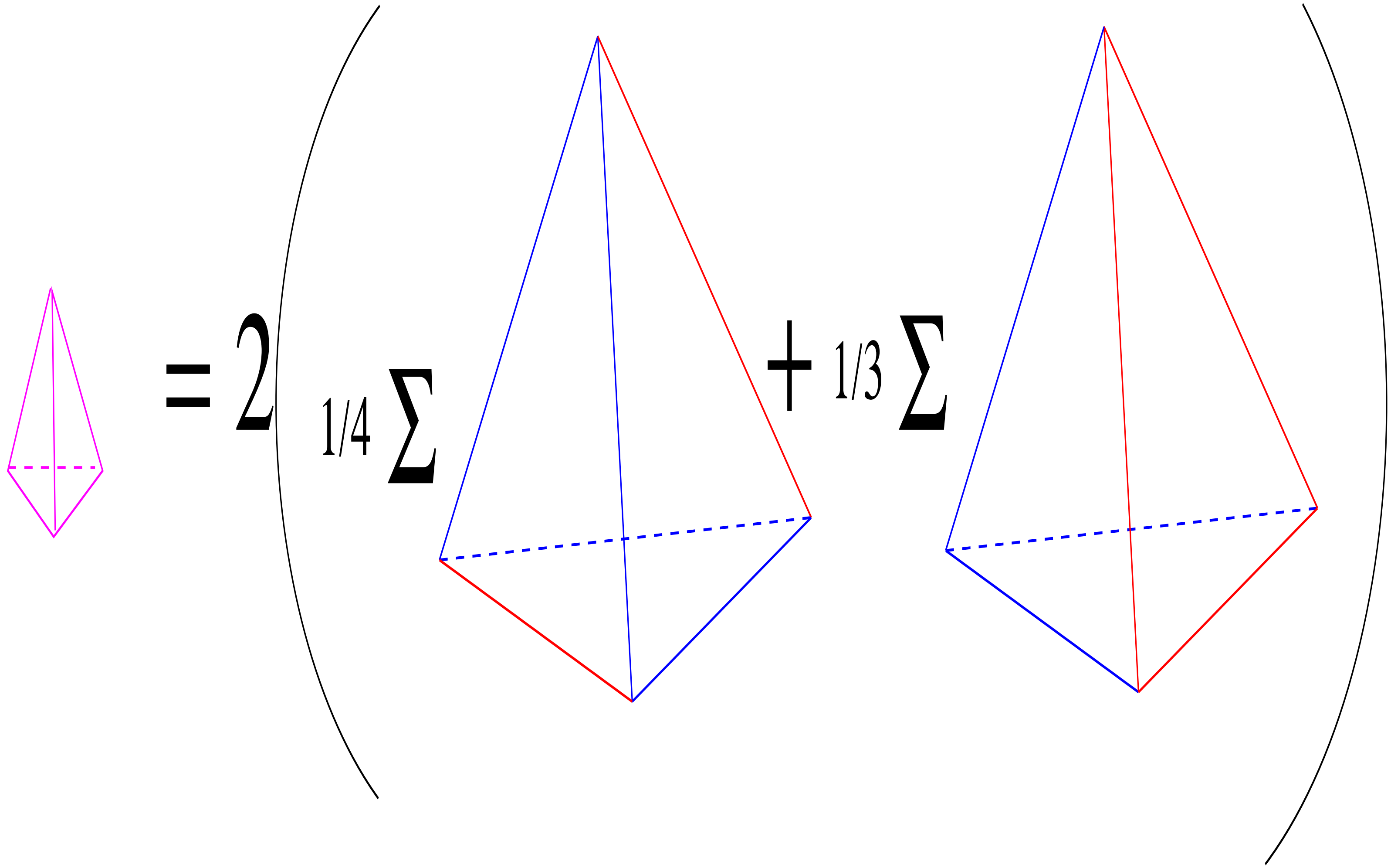
$\sum$



$= 4(ad+be+cf)$

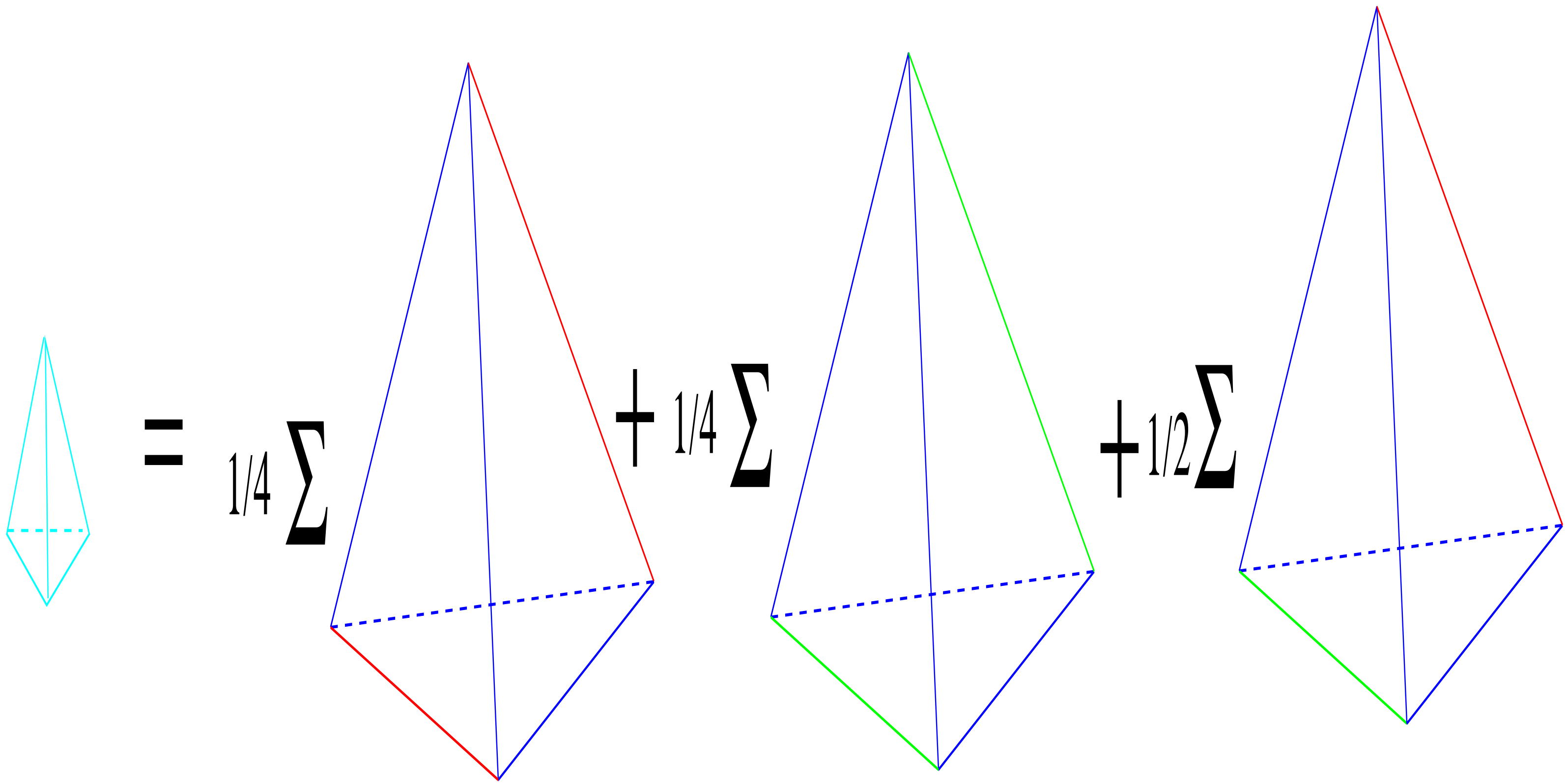
Notation

Using this notation, we define sum need combinations of such monomials. One is...



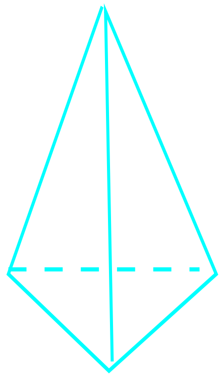
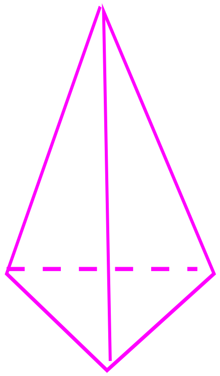
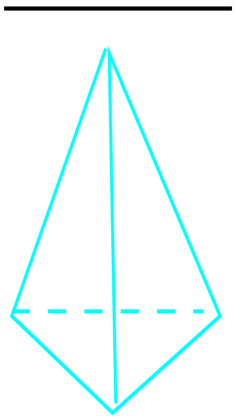
Notation

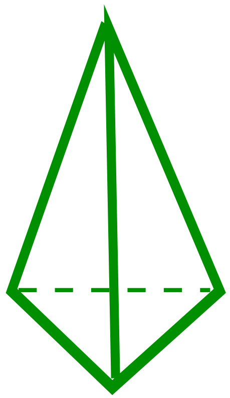
and another is below. Note these expressions are invariant under tetrahedral symmetries.



Notation

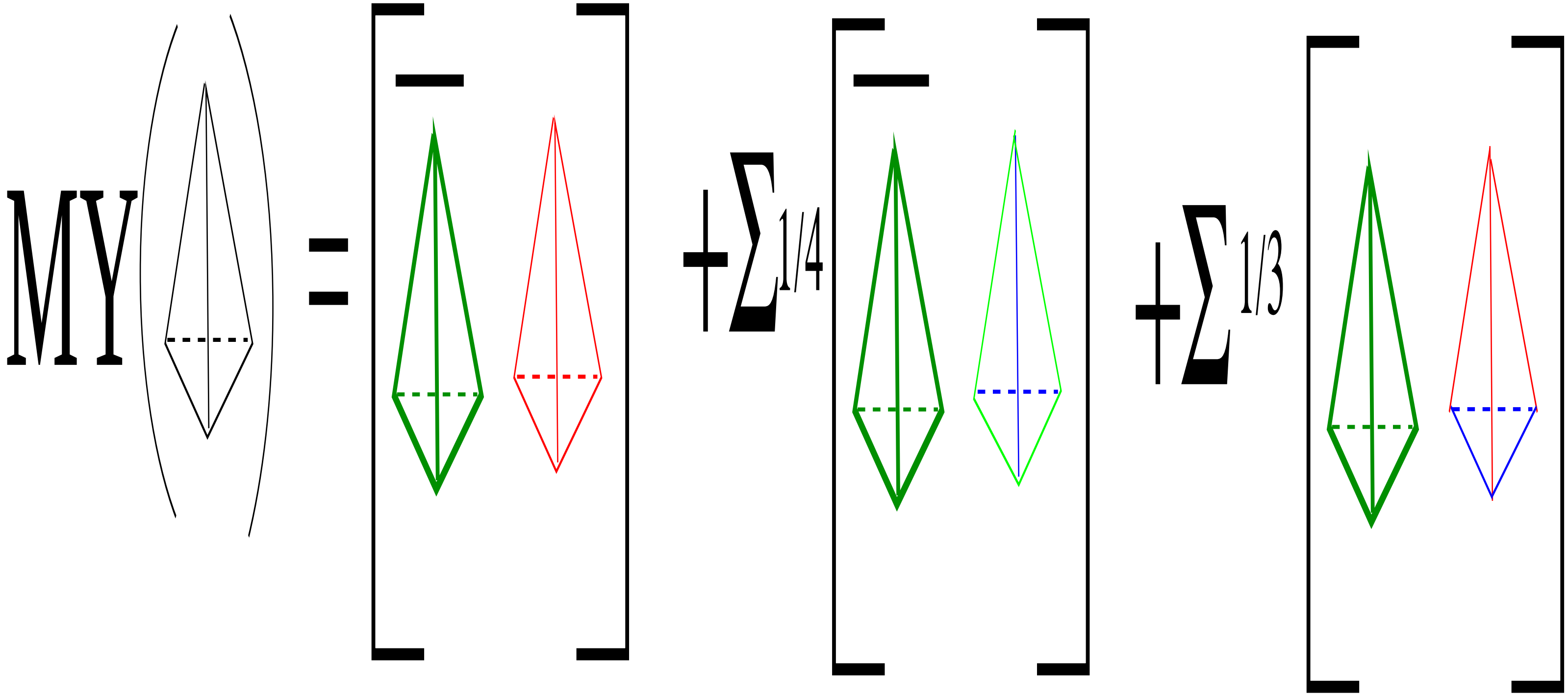
Using these expressions, we form a quadratic polynomial and fix one of its roots.

h(x)=  x^2 +  x + 

a root: 

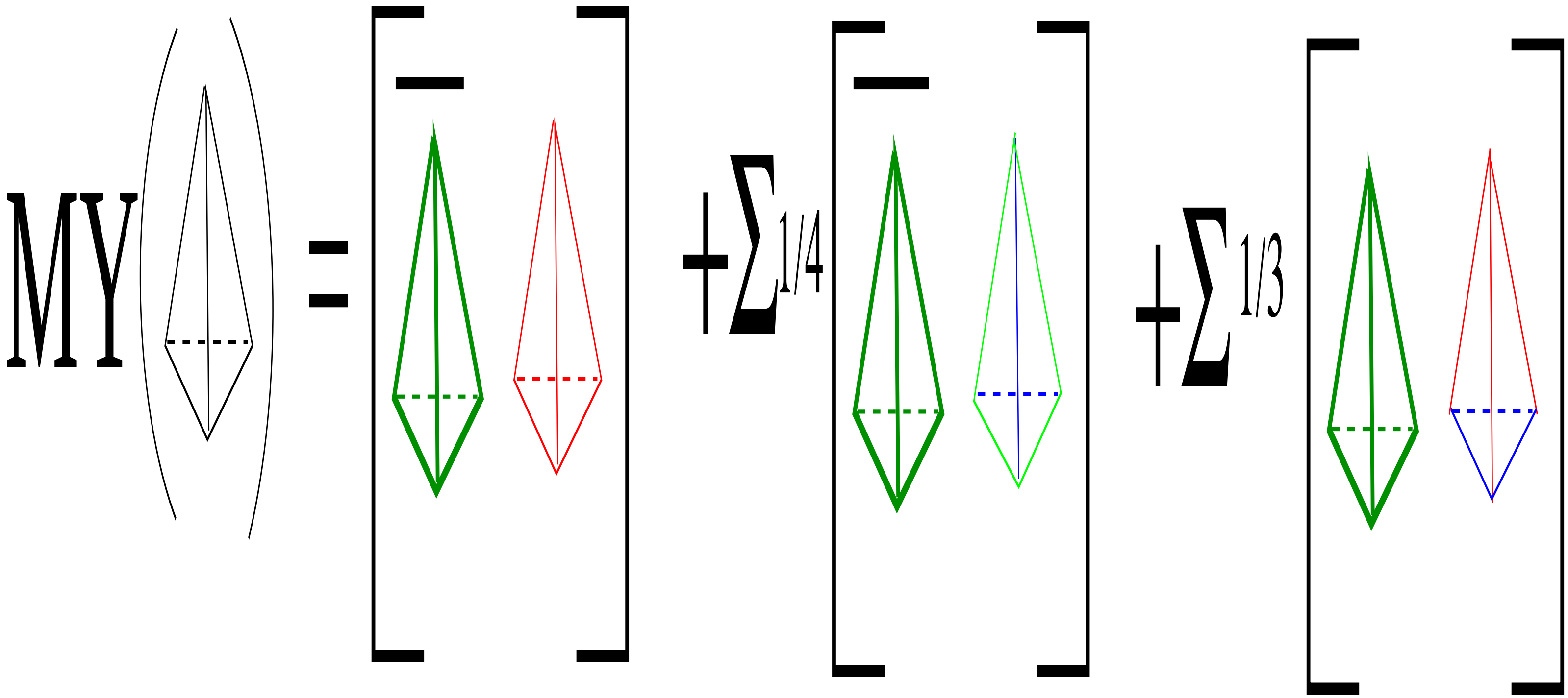
The Fundamental Scissors' Class

Using this root we form the following scissors class consisting of 8 ideal tetrahedra.



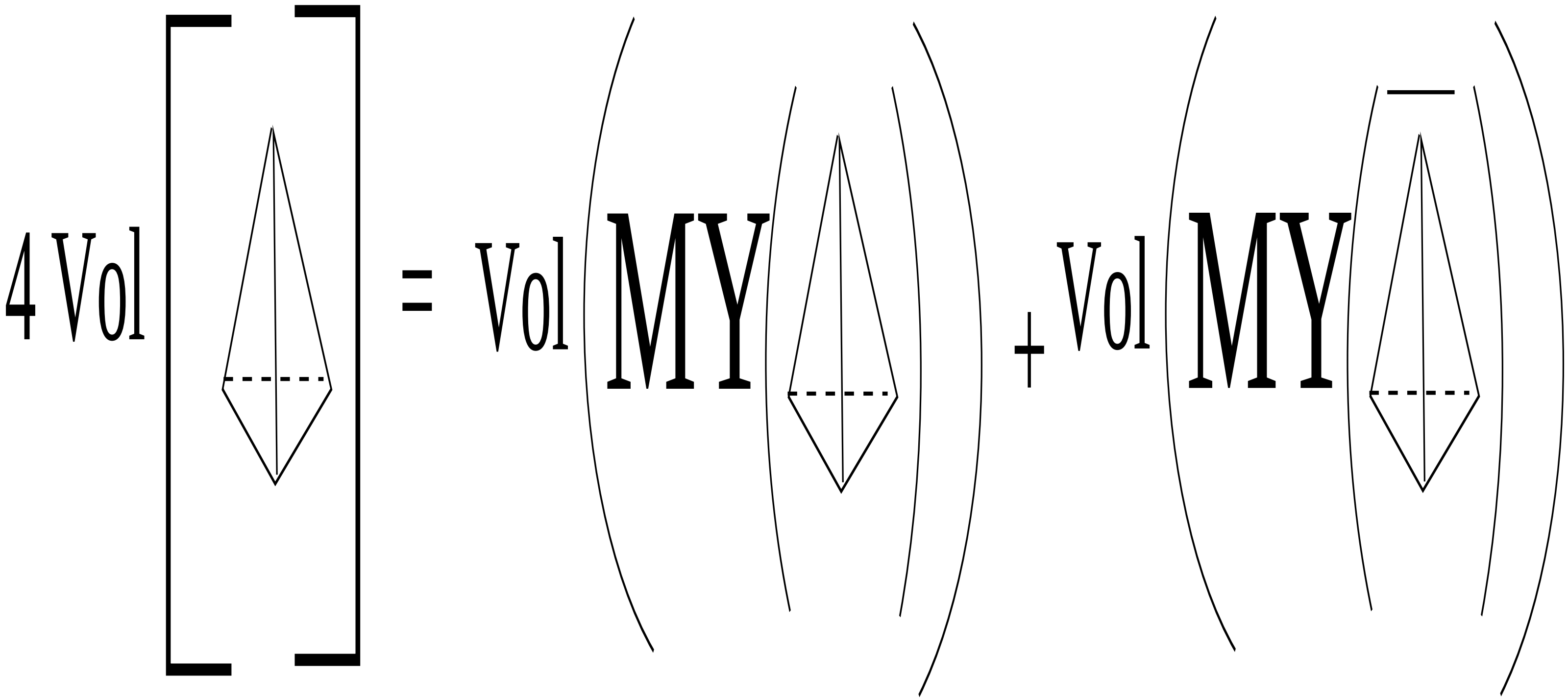
The Fundamental Scissors' Class

In fact, all these complex numbers are unit size so these correspond to 8 isosceles ideal tetrahedra.



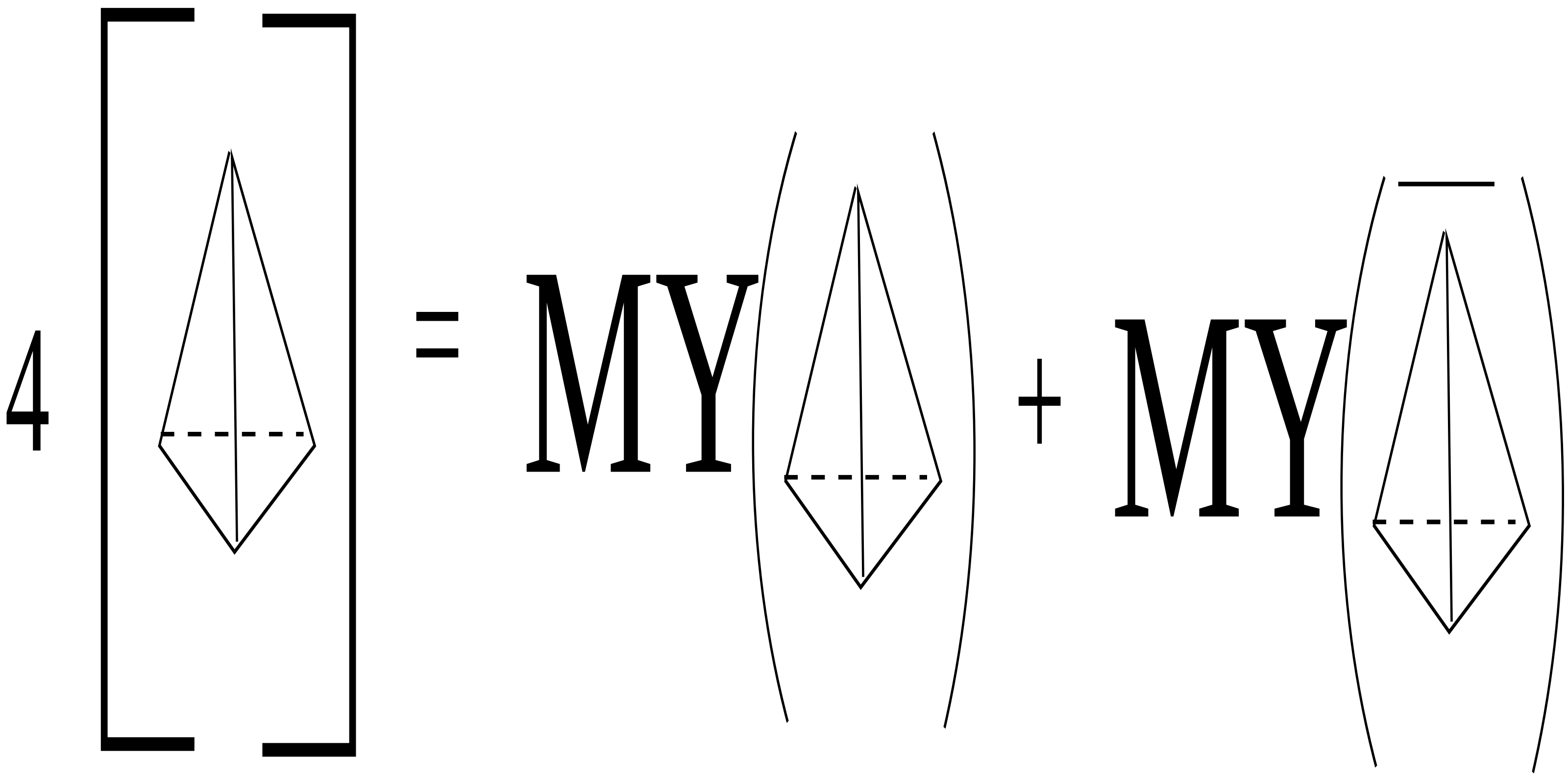
The Volume Theorem

Theorem (J. Murakami, M. Yano)



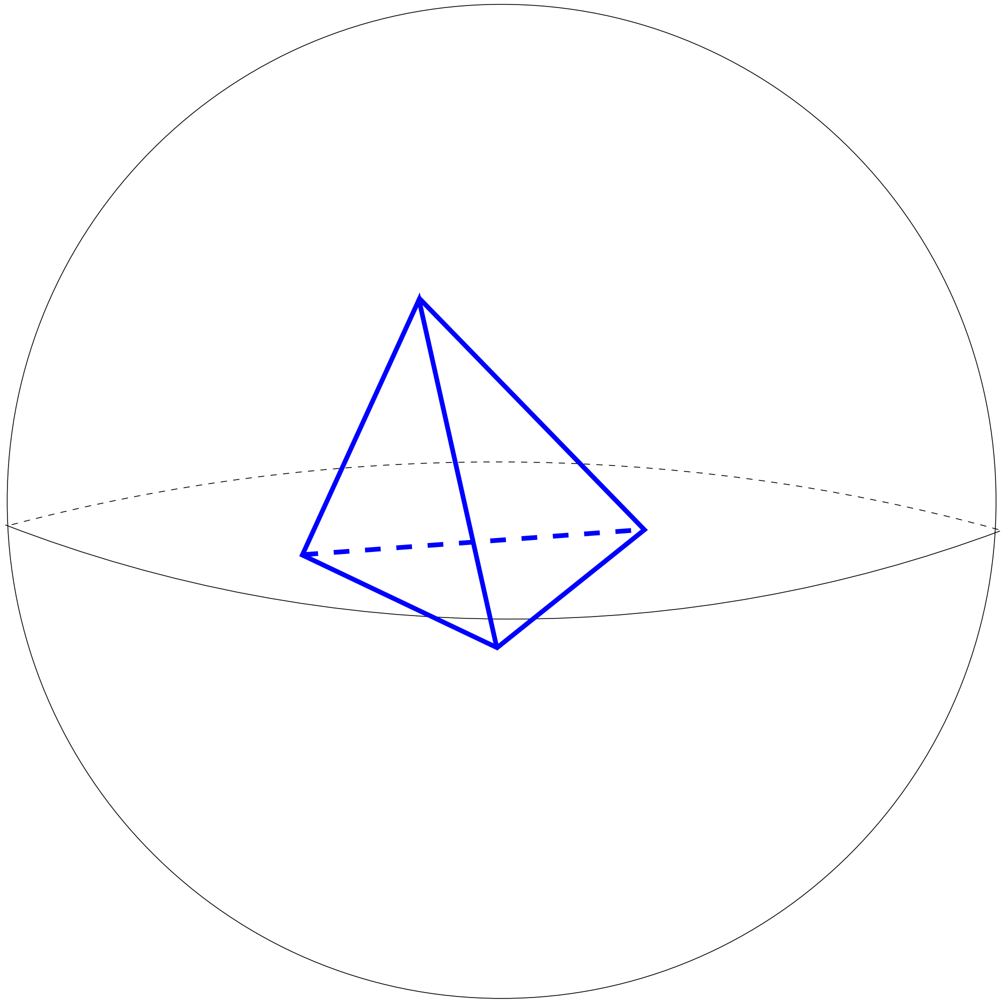
The Scissors Theorem

Theorem (Doyle,Leibon)



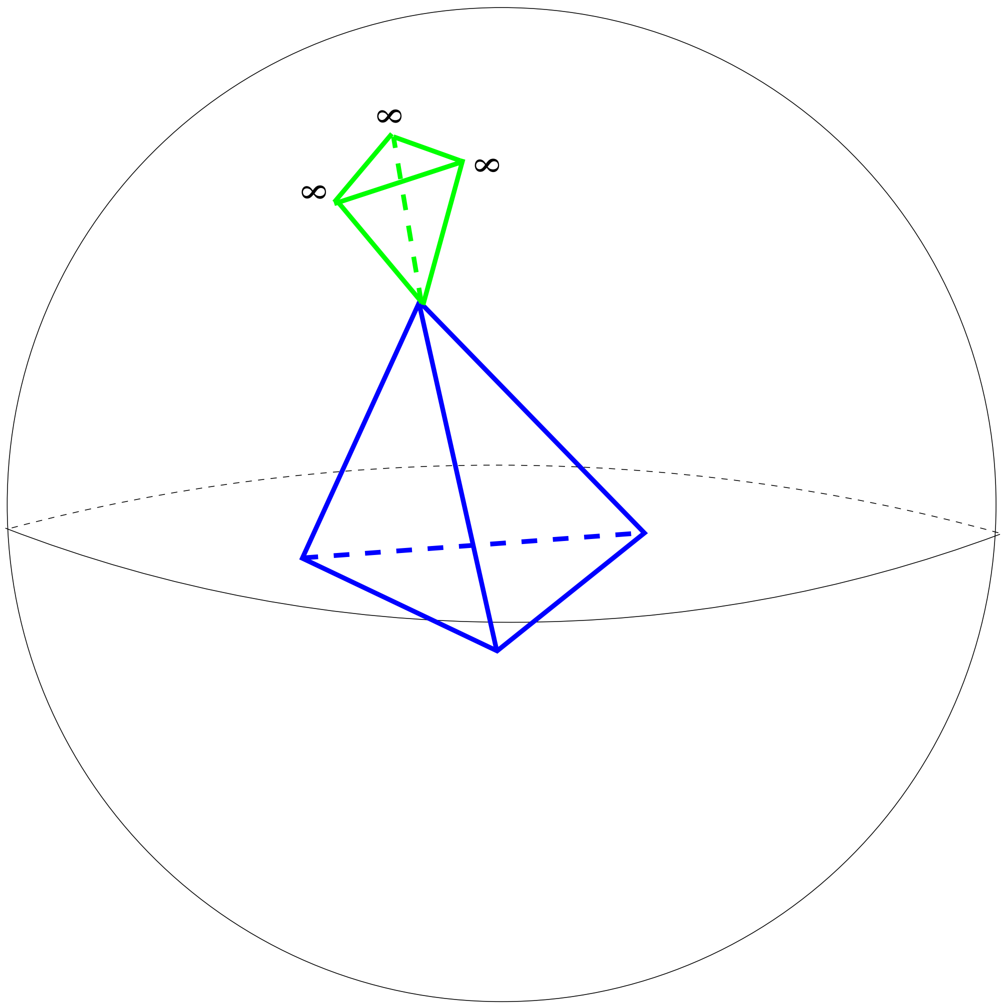
The Supertetrahedron

To prove this theorem, we start with a finite tetrahedron.



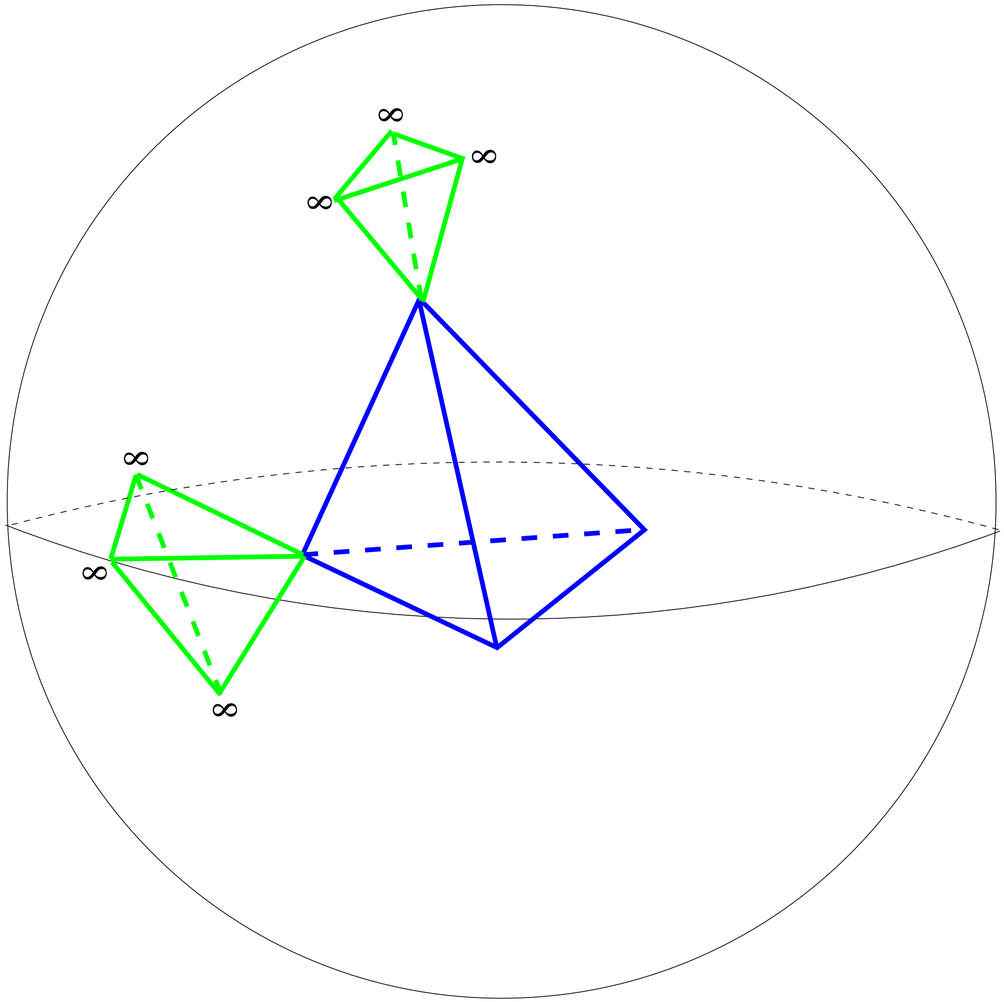
The Supertetrahedron

Adjoin to each vertex its associated "twist tie".



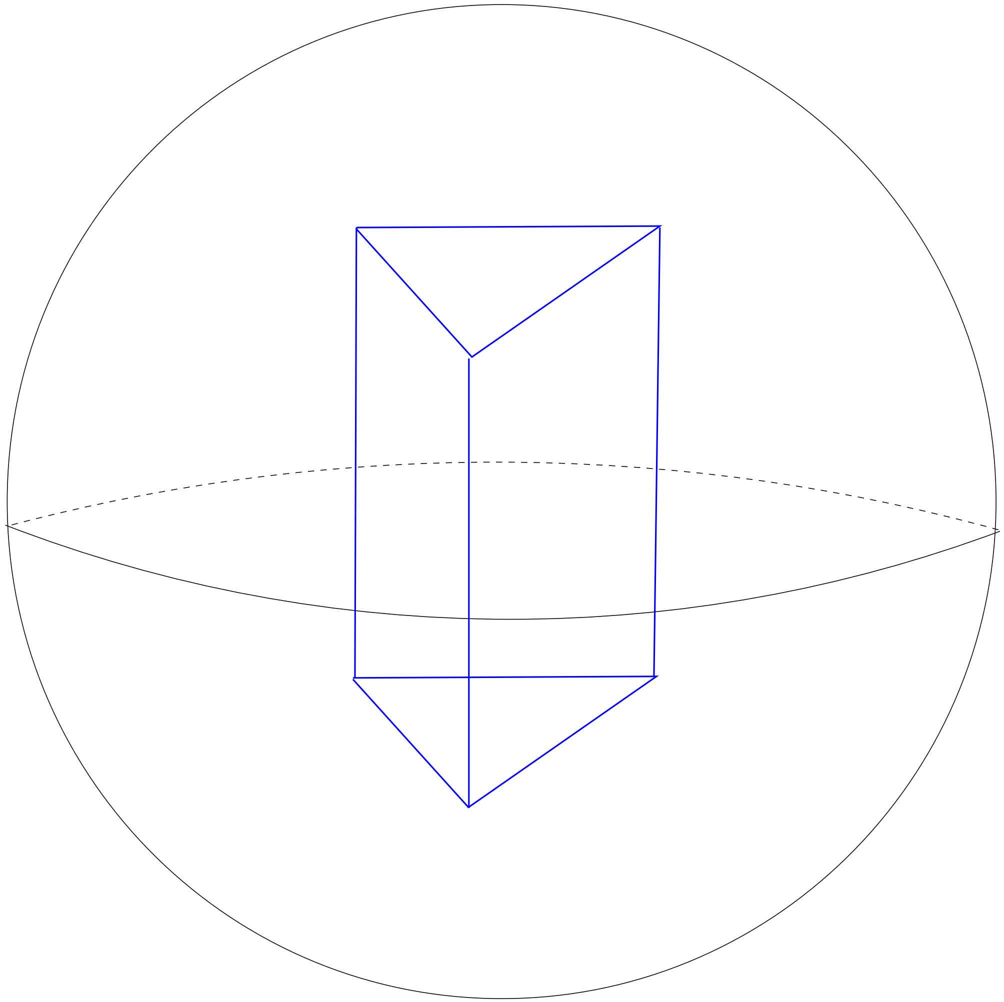
The Supertetrahedron

This is really the adjoining of half of a "twisted prism".



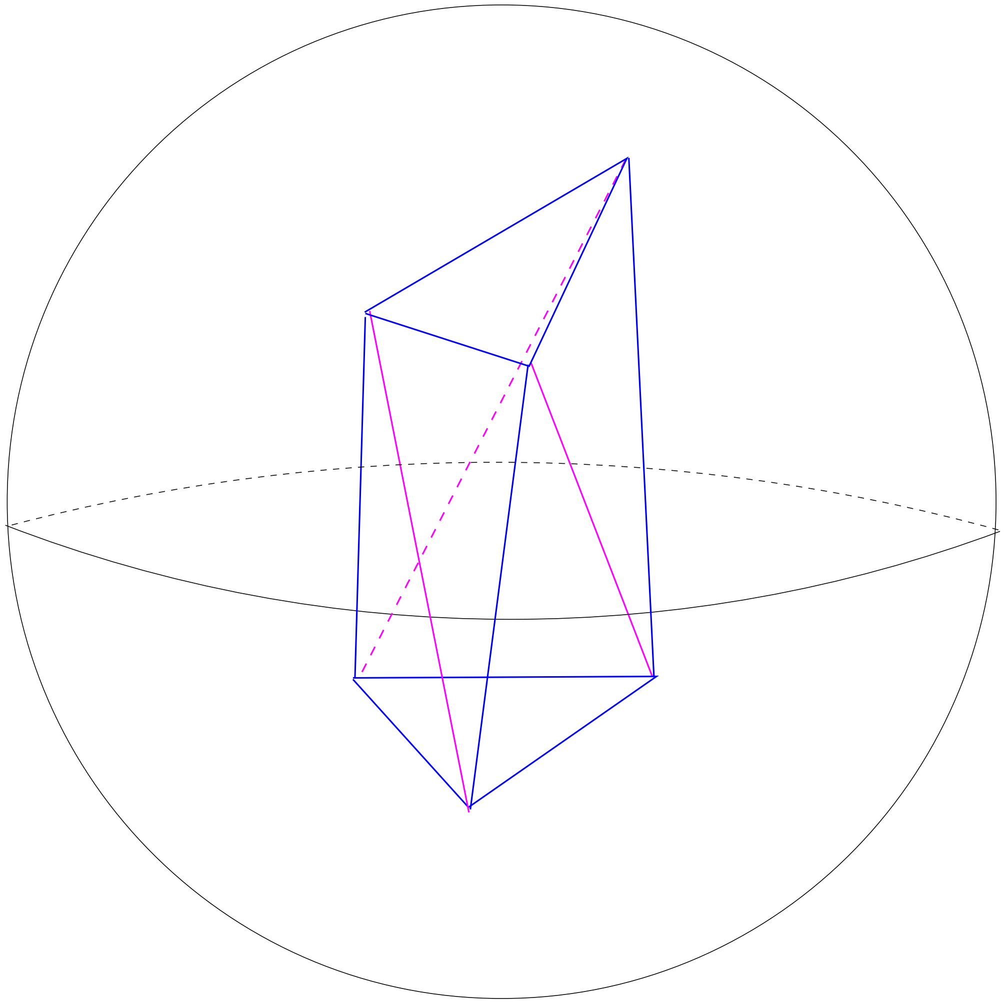
The Convex Prism

Namely here is untwisted prism. We twist...



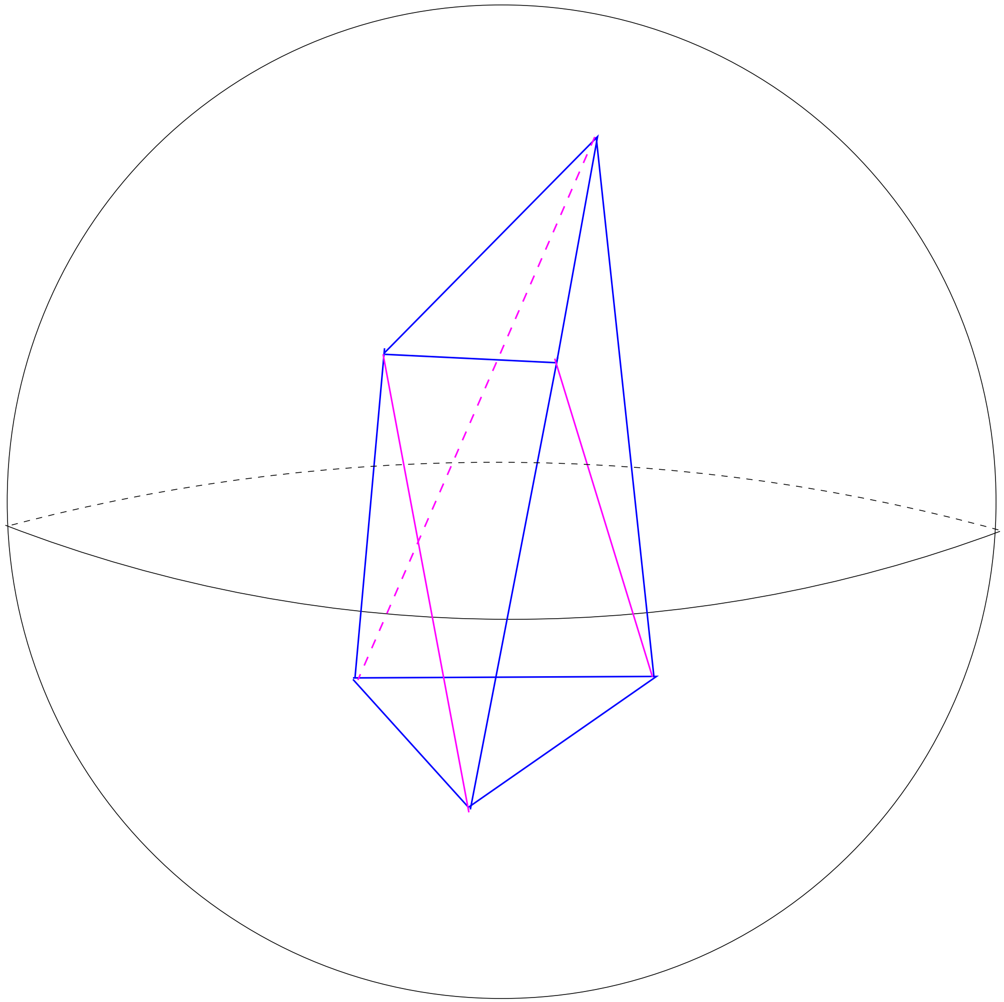
Twisting the prism.

until..



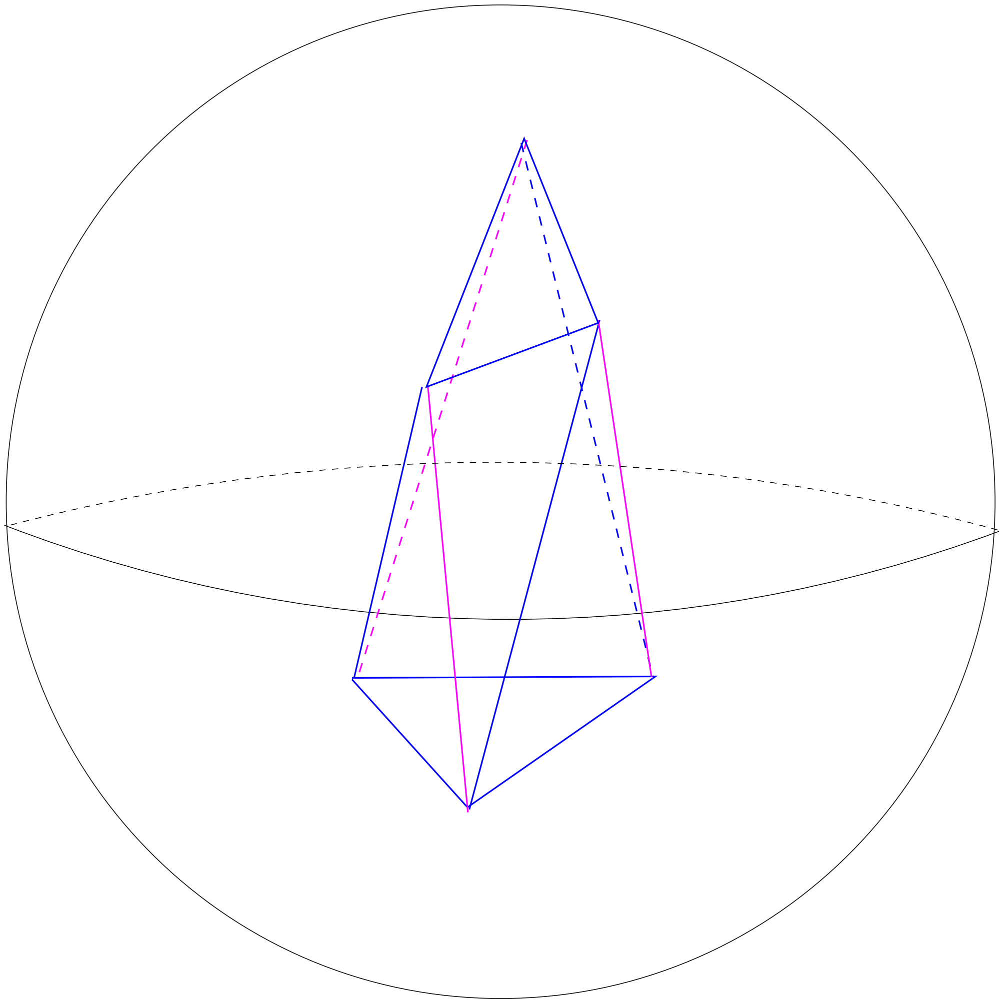
Twisting the prism.

until until..



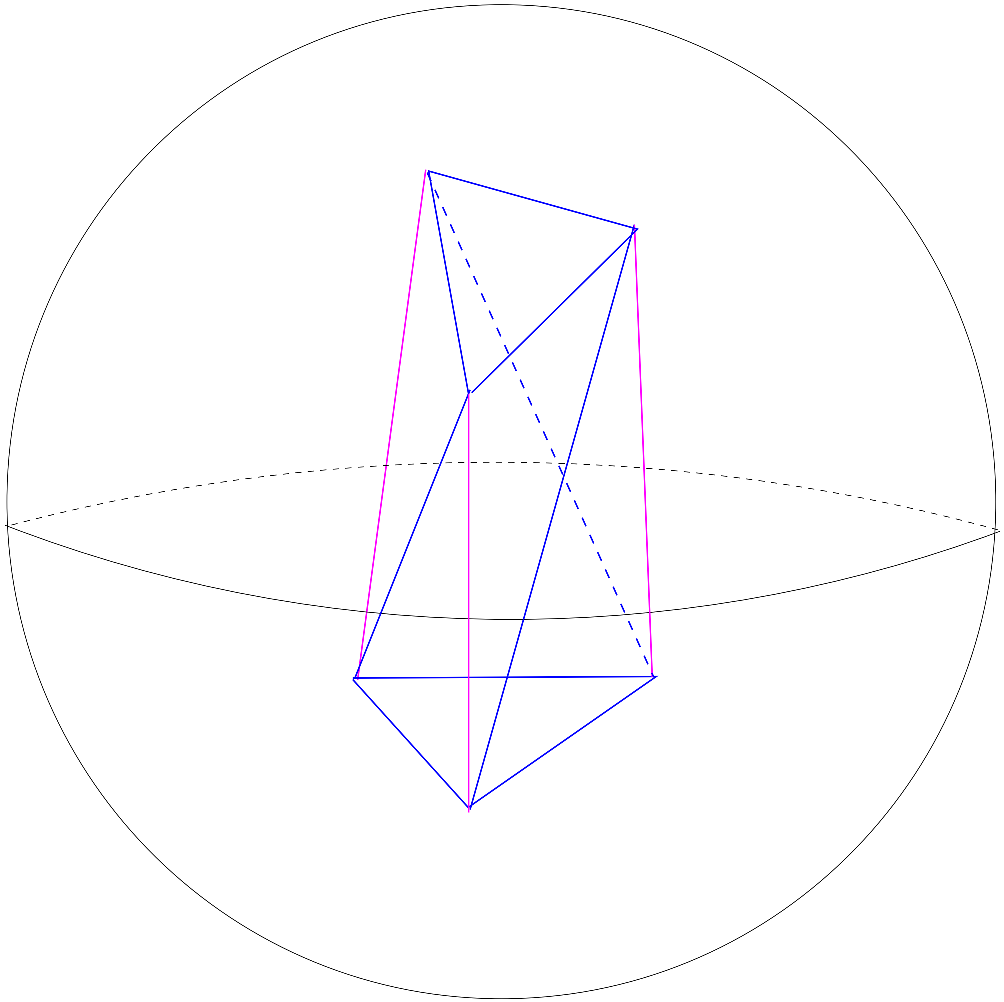
Twisting the prism.

until..



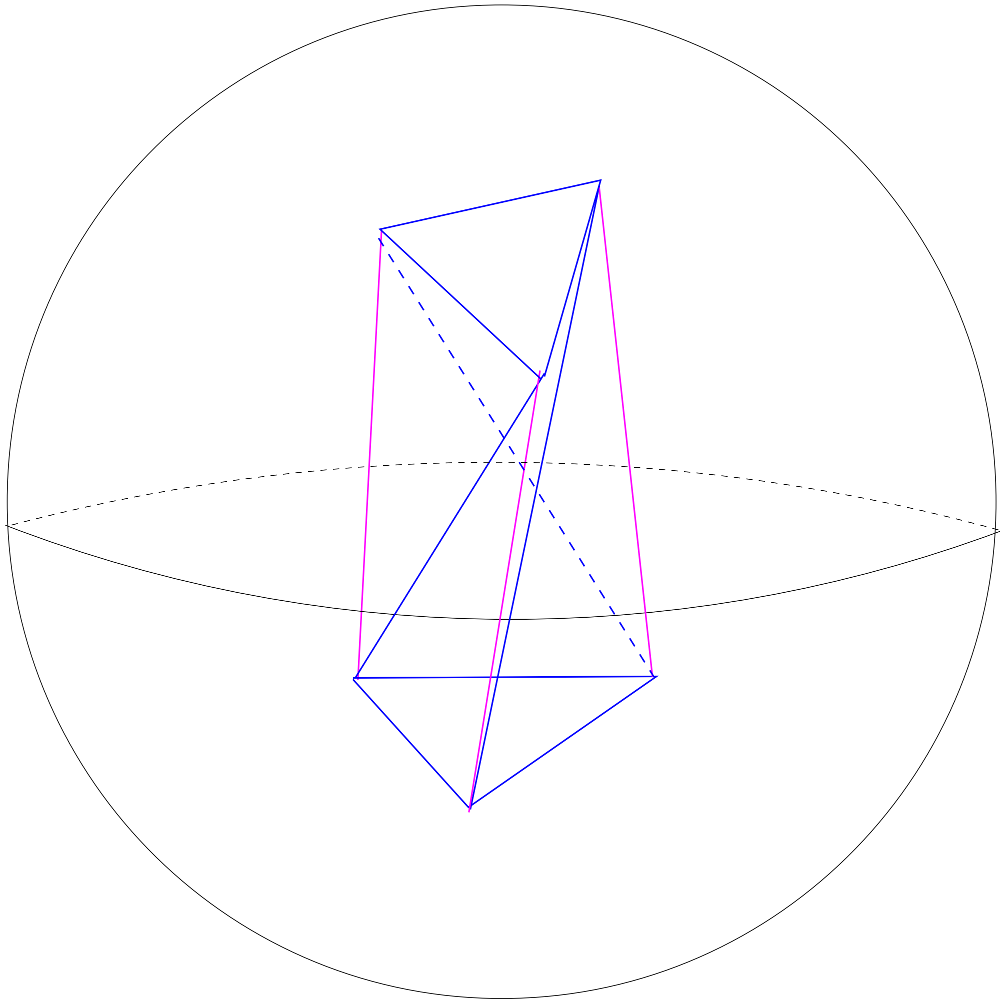
Twisting the prism.

until..



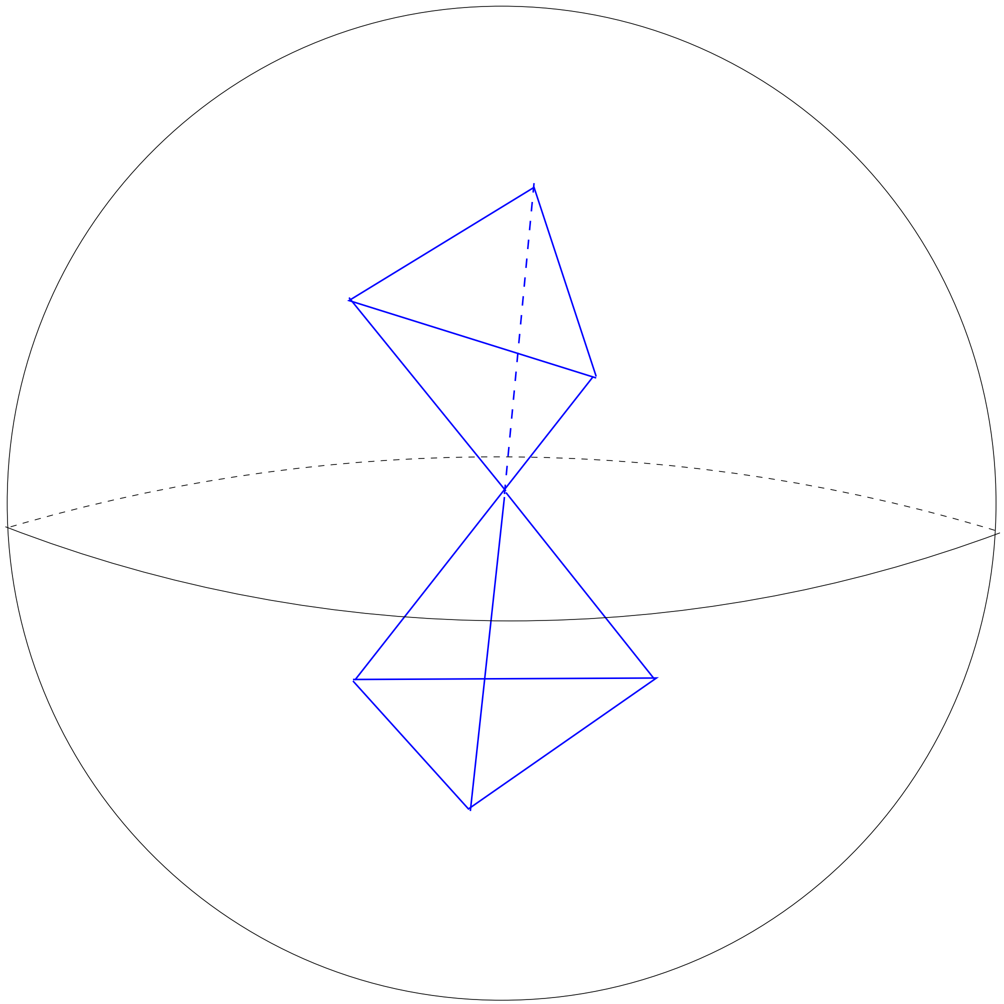
Twisting the prism.

until..



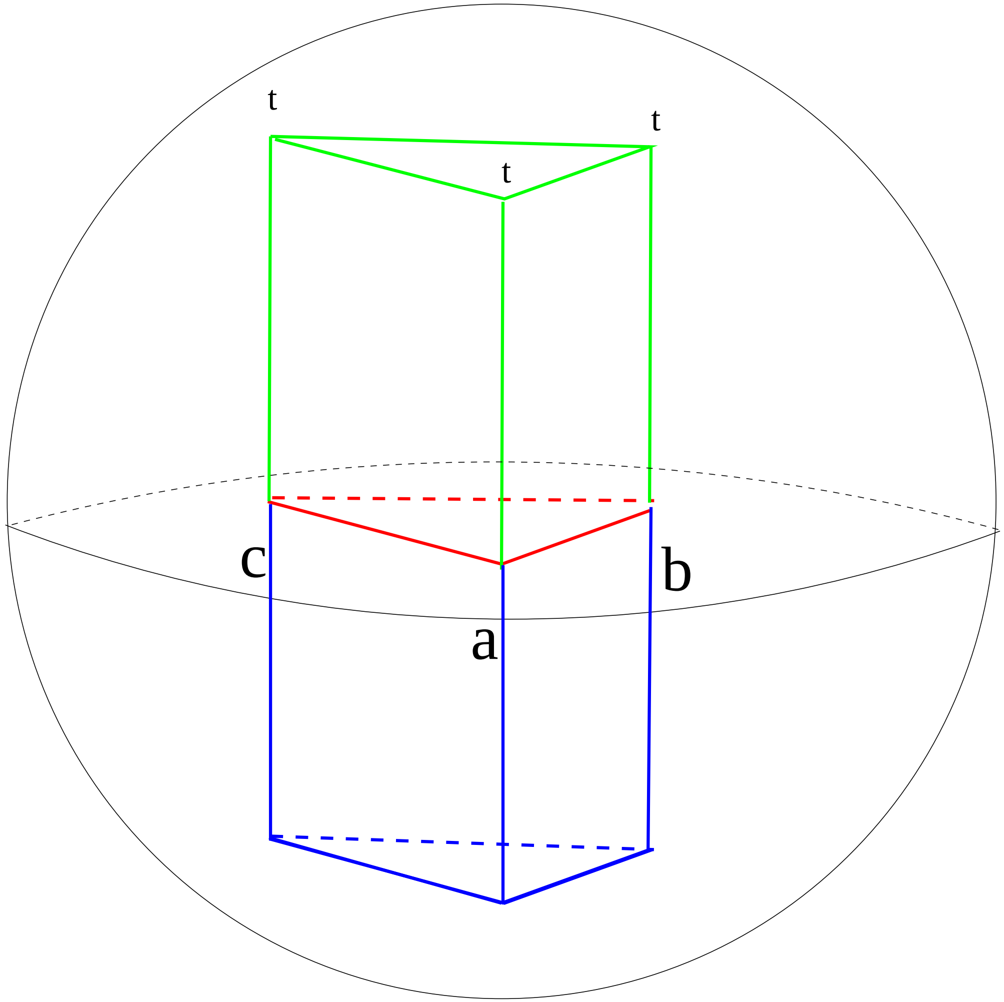
The Twisted Prism.

we've gone half way around.



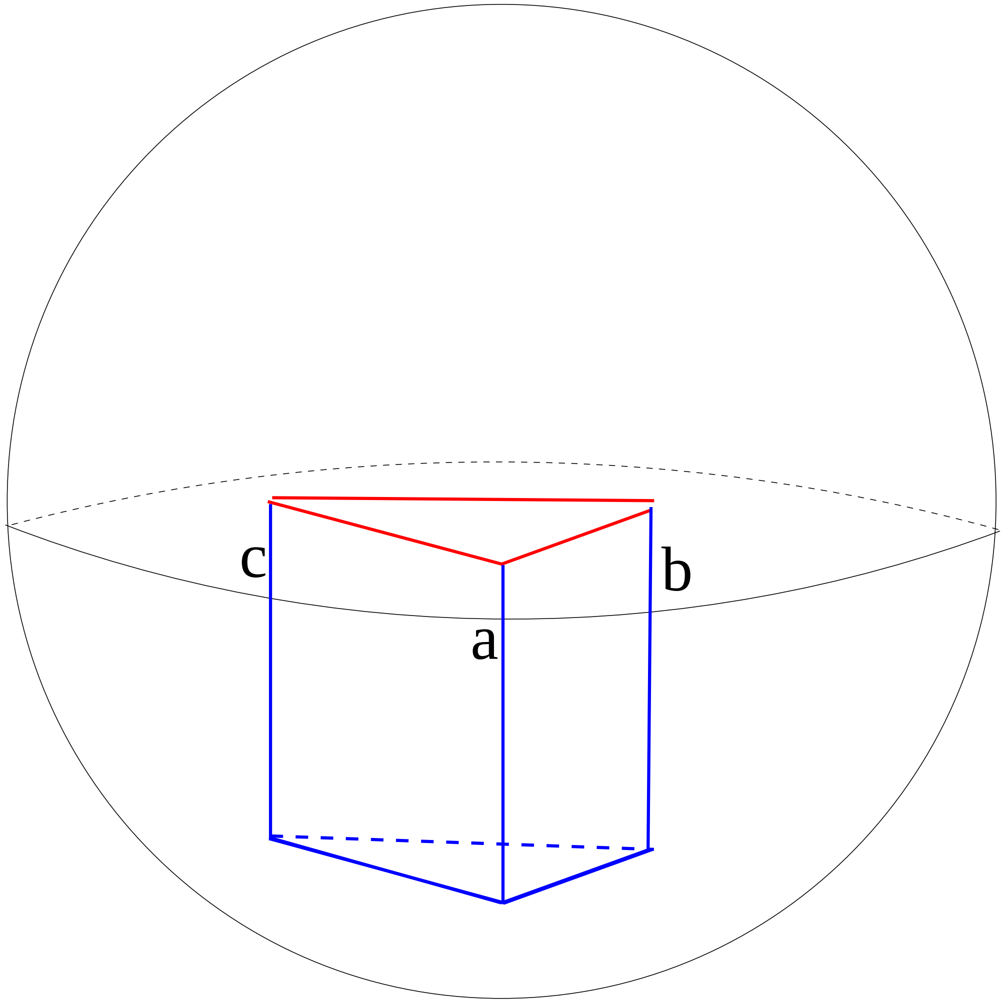
Prism Symmetry

Every prism is symmetric, with its top being the positively oriented mirror image of the bottom.



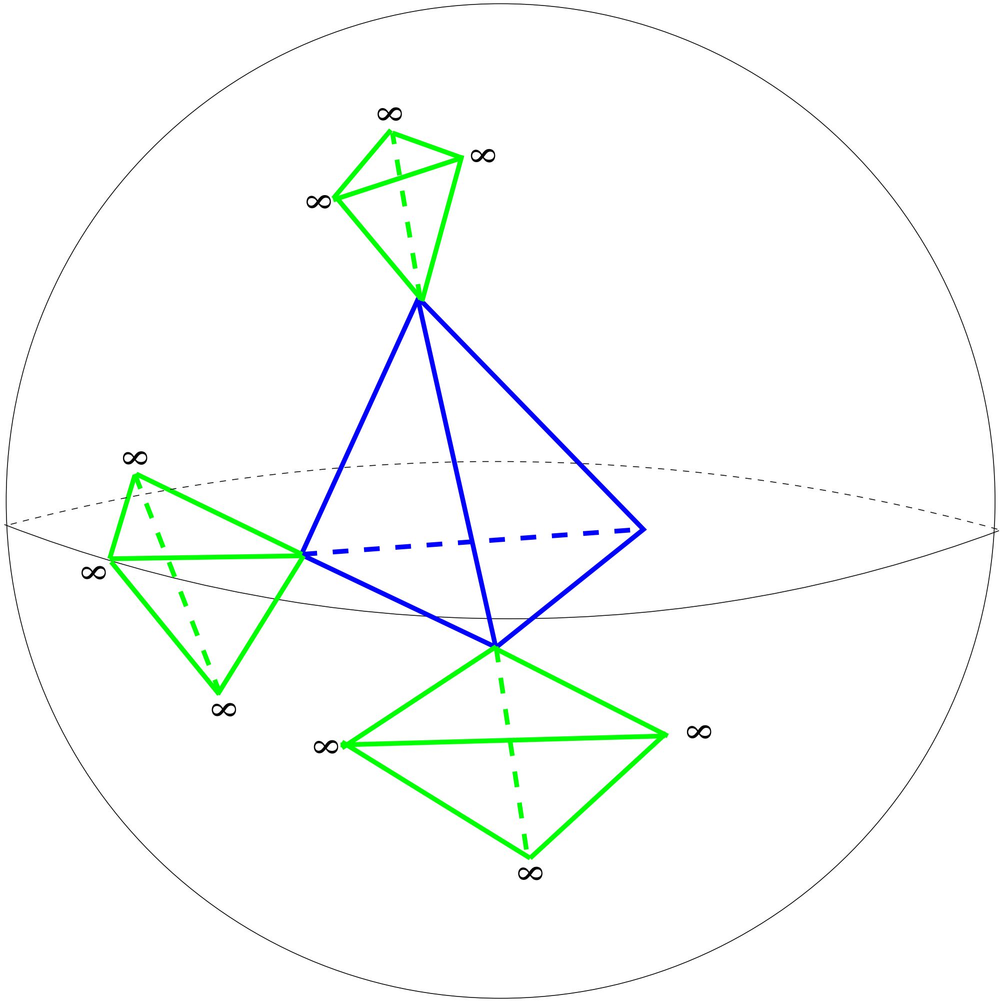
Prism Symmetry

We can always cut off the top.



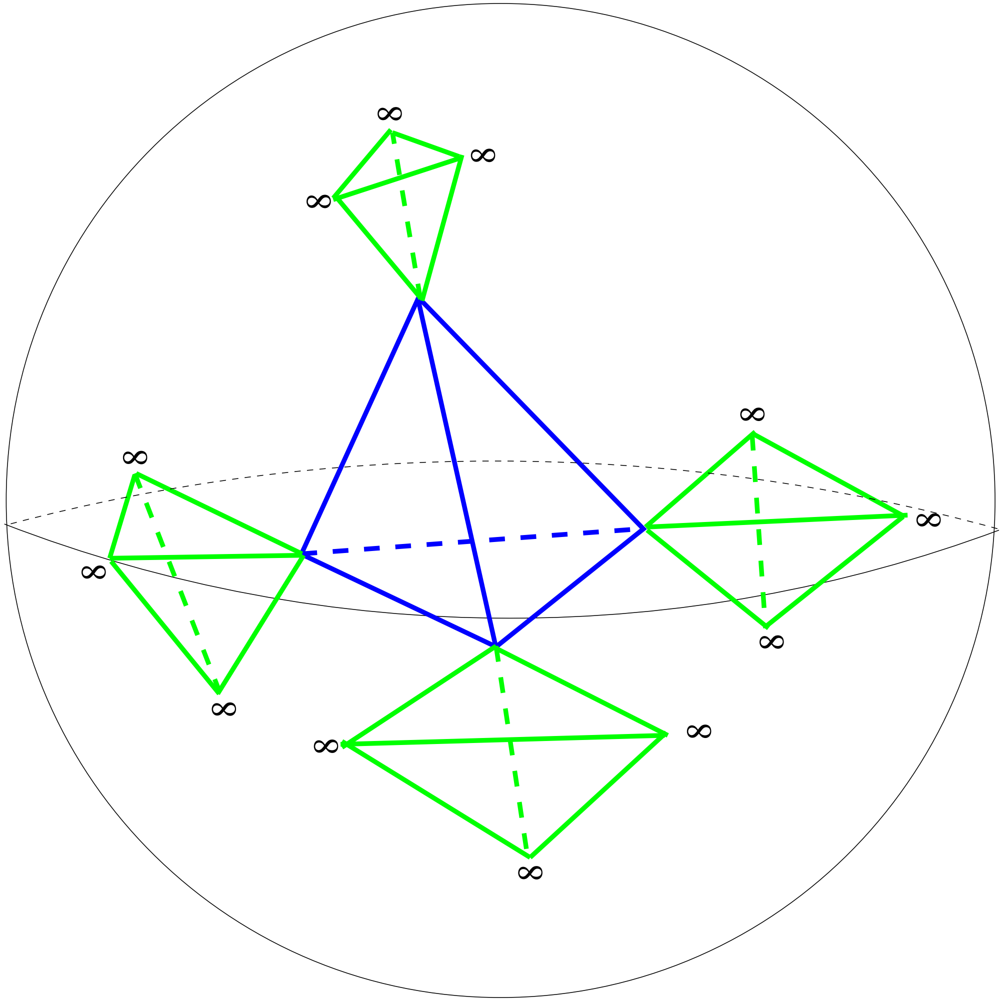
The Supertetrahedron

Back to adjoining our twist ties.



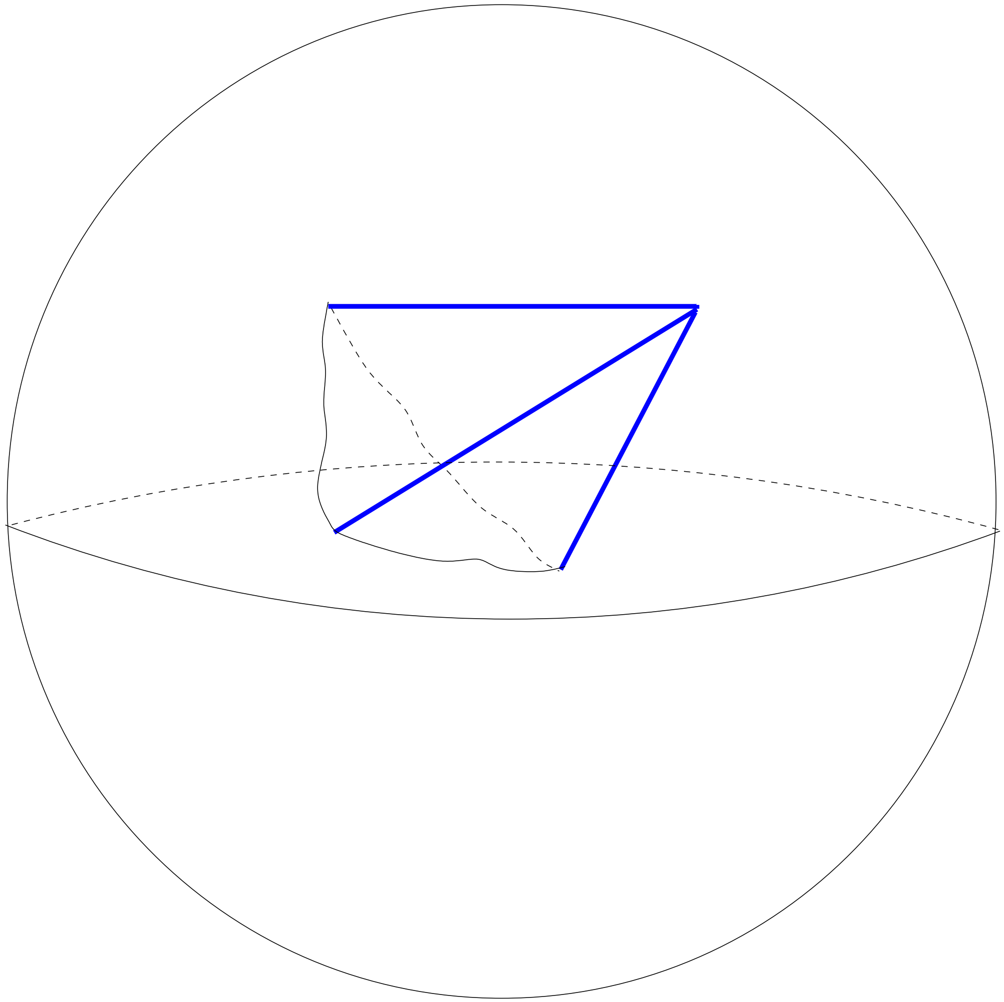
The Supertetrahedron

This is a *supertetrahedron*.



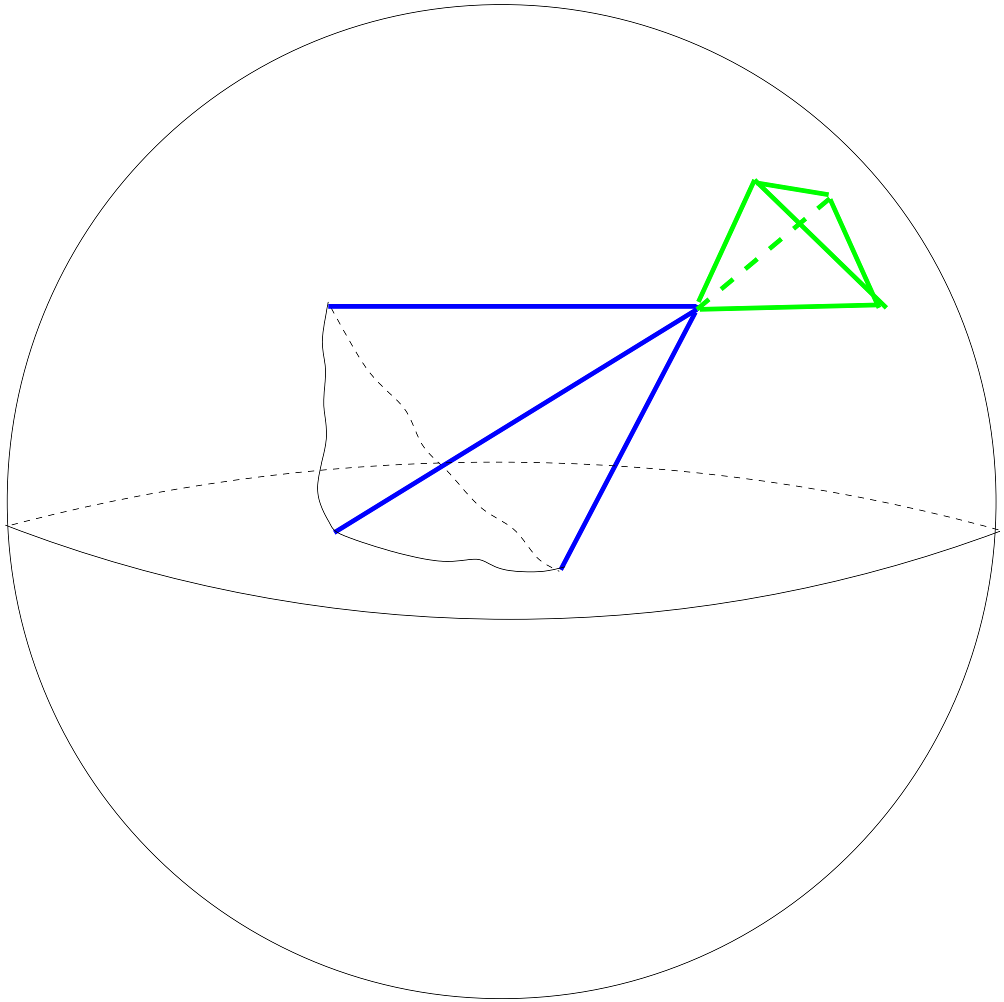
Analytic Continuation

Now we will reconsider the twisting process. Here we see a vertex of a finite polyhedron.



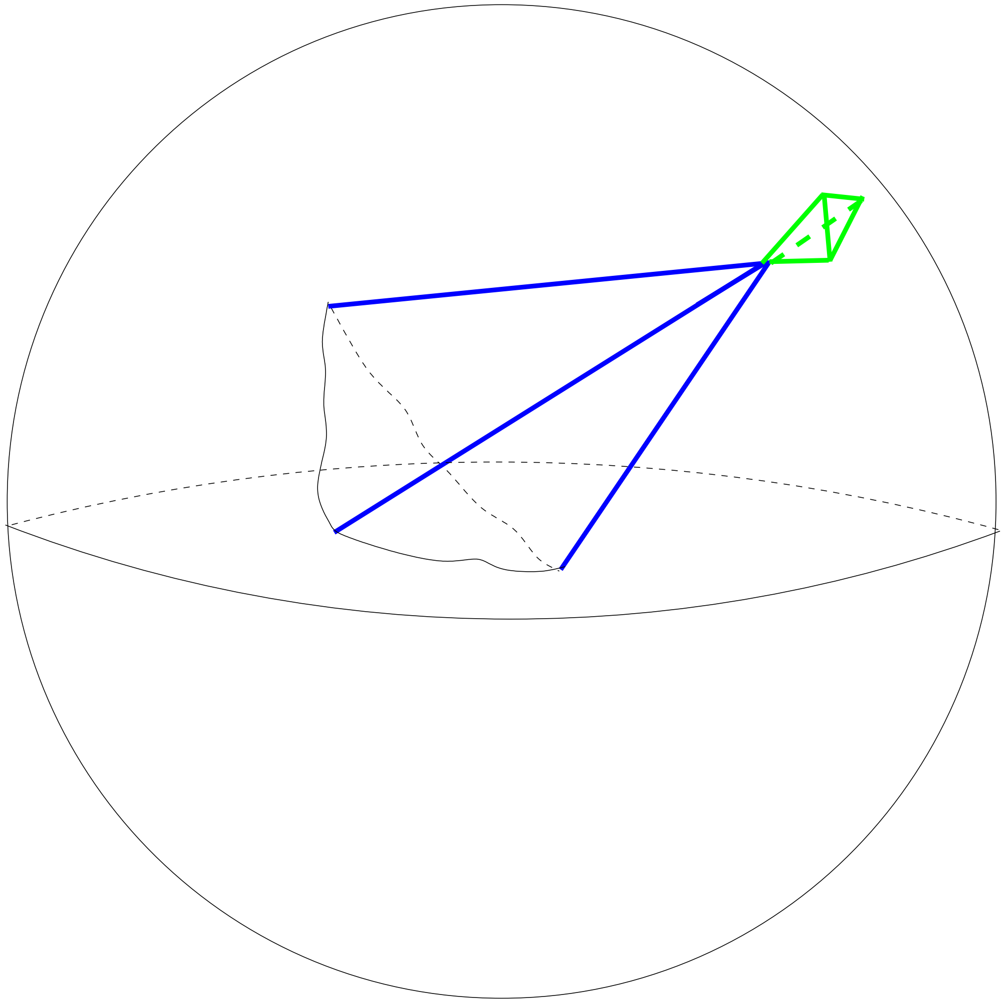
Analytic Continuation

Let us adjoin its associated twist tie.



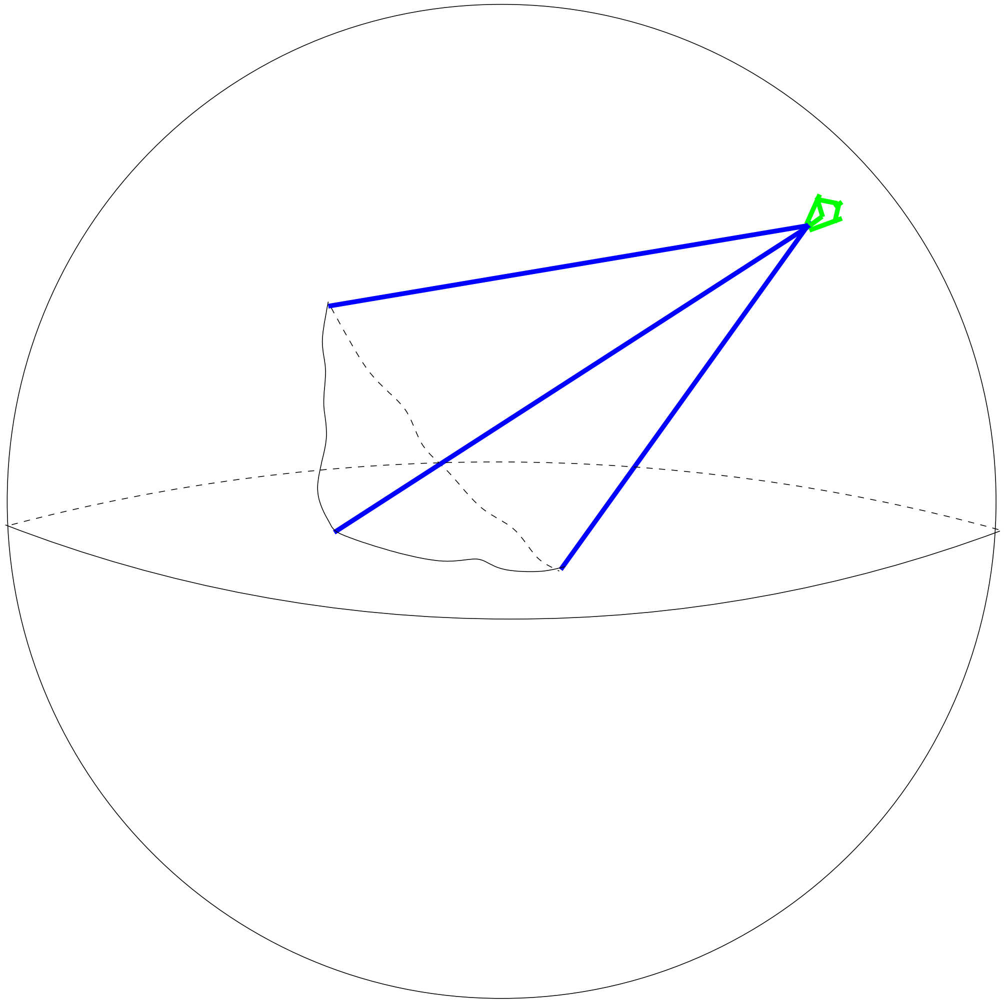
Analytic Continuation

Our goal is to "analytically continue" in order find a better view of a finite polyhedron.



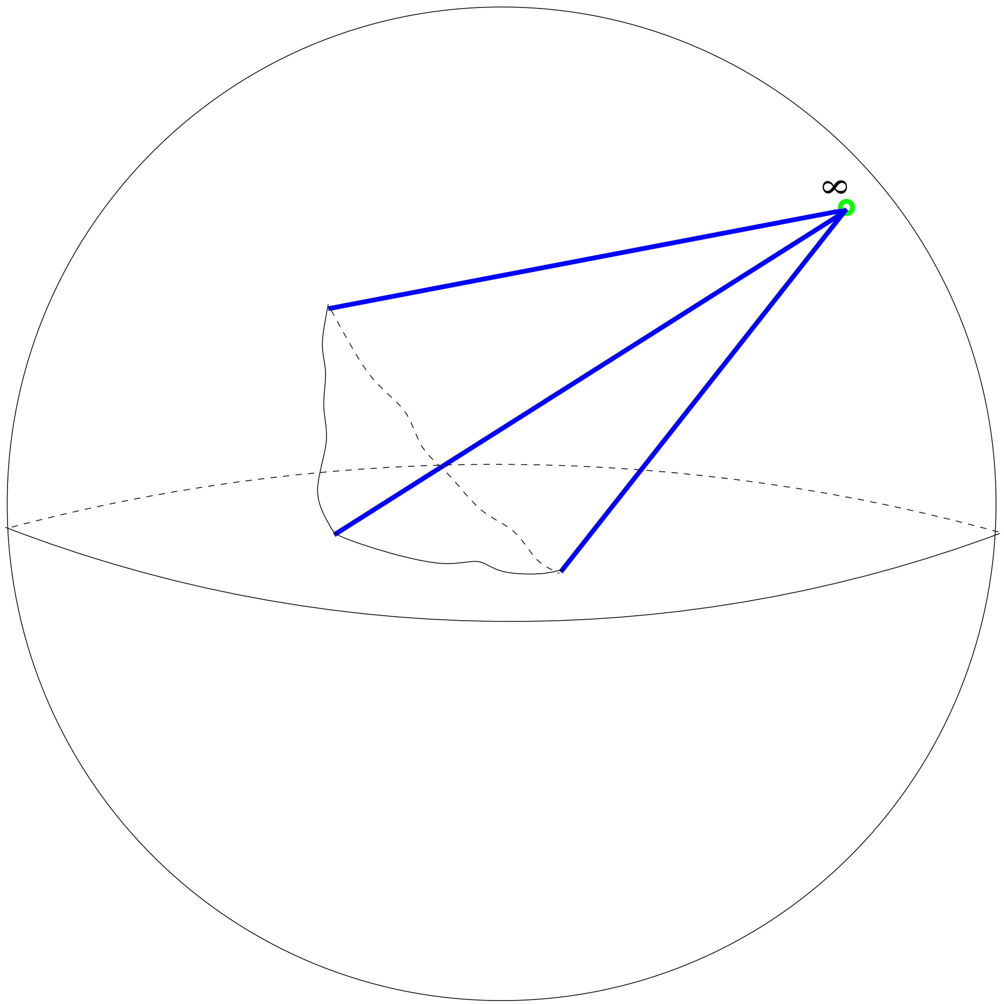
Analytic Continuation

To do this, we will push the finite vertex beyond infinity.



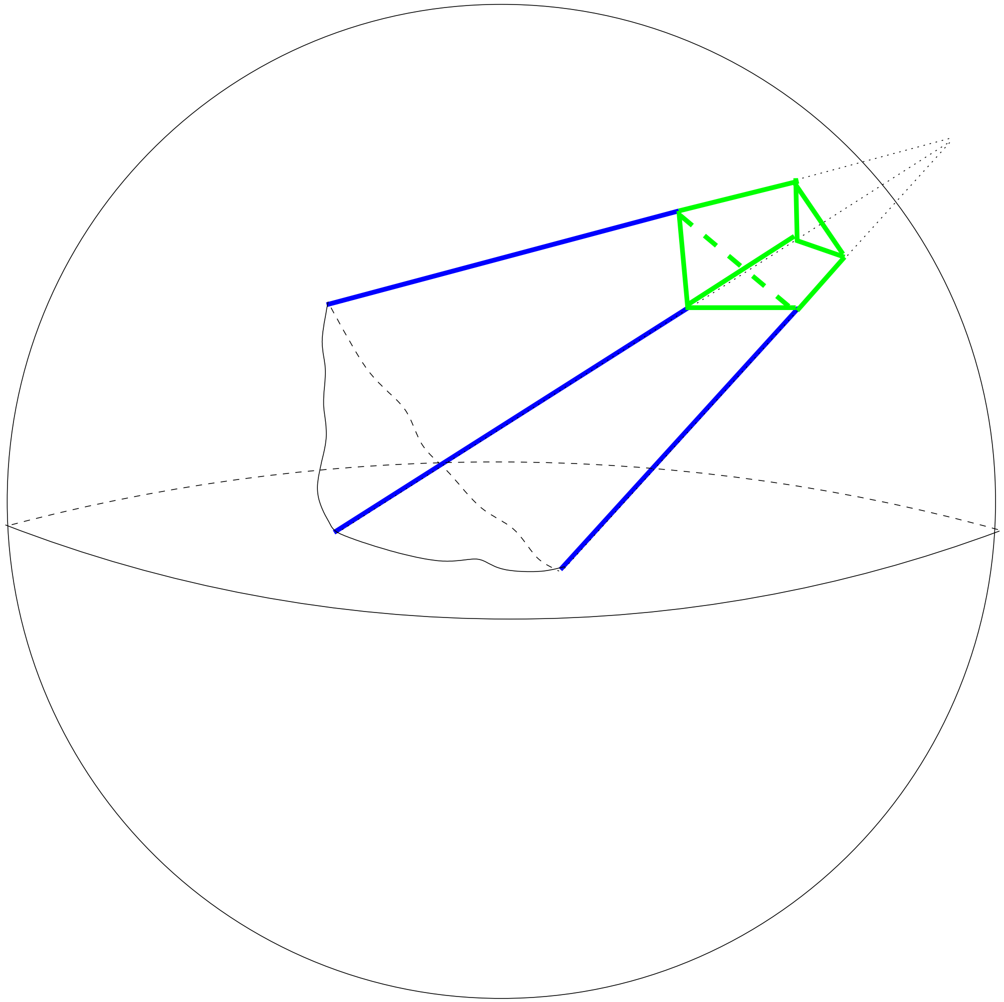
Analytic Continuation

At this point we see an idea vertex. Really at this ideal vertex there is a "hidden ideal tetrahedron".



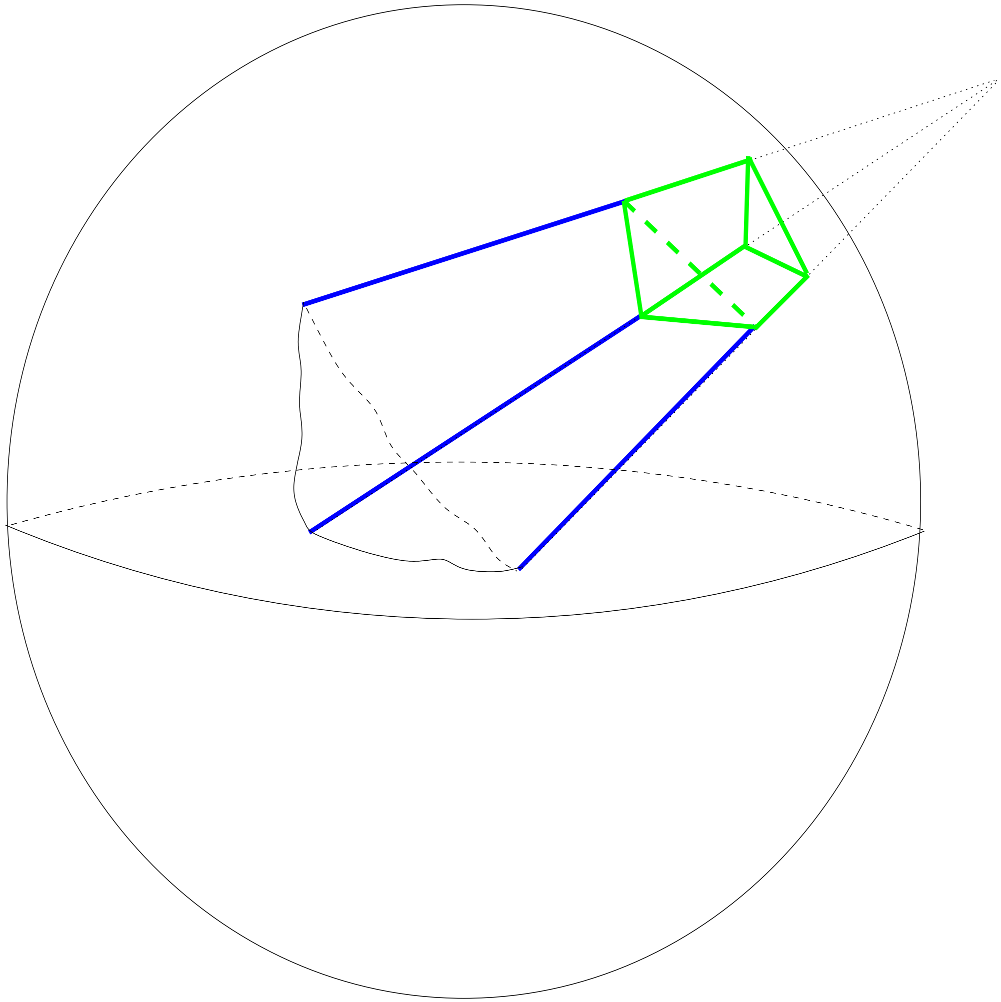
Analytic Continuation

Now we are beyond infinity. Notice we have untwisted our twist tie.



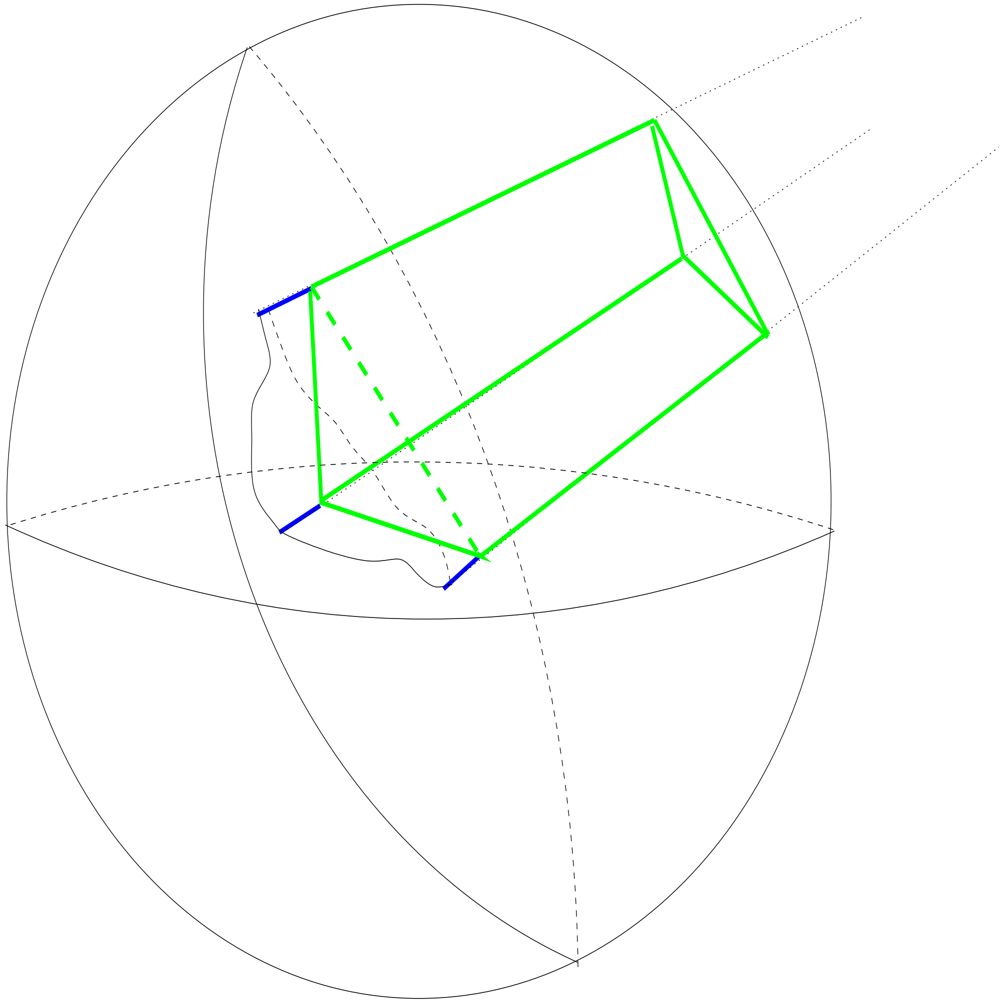
Analytic Continuation

The green part is constructed via the polar plane of the ideal point. This region is the top half of a prism.



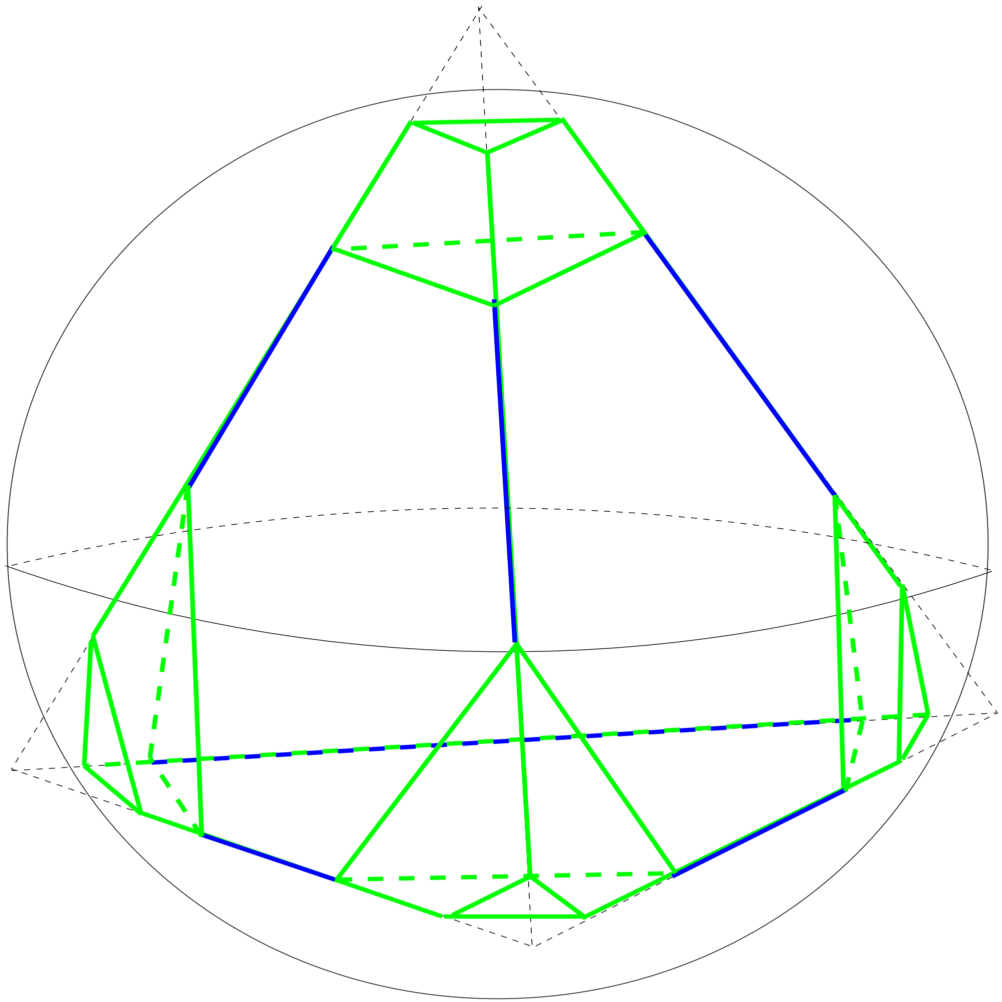
Analytic Continuation

We can move this hyperideal vertex to infinity in the Klein model.



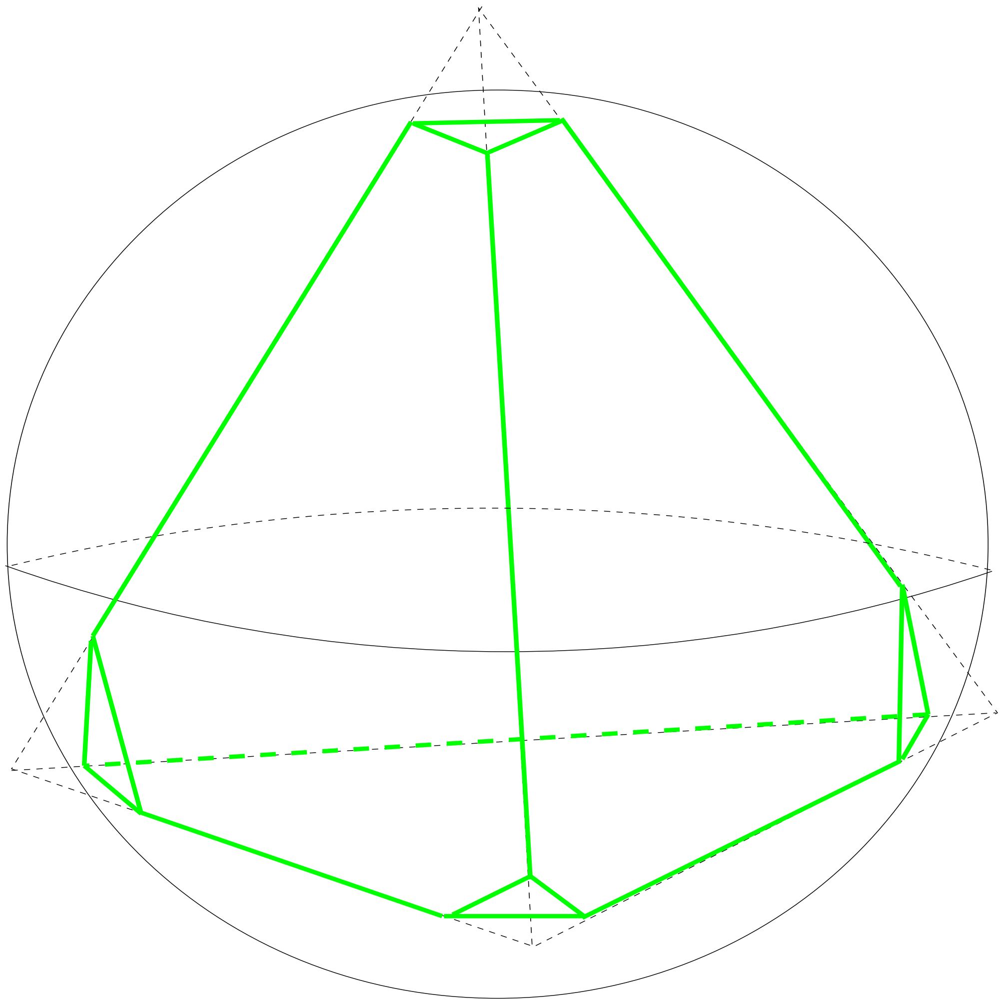
The Convex Supertetrahedron

Applying analytic continuation to our finite supertetrahedron results in the *convex supertetrahedron*.



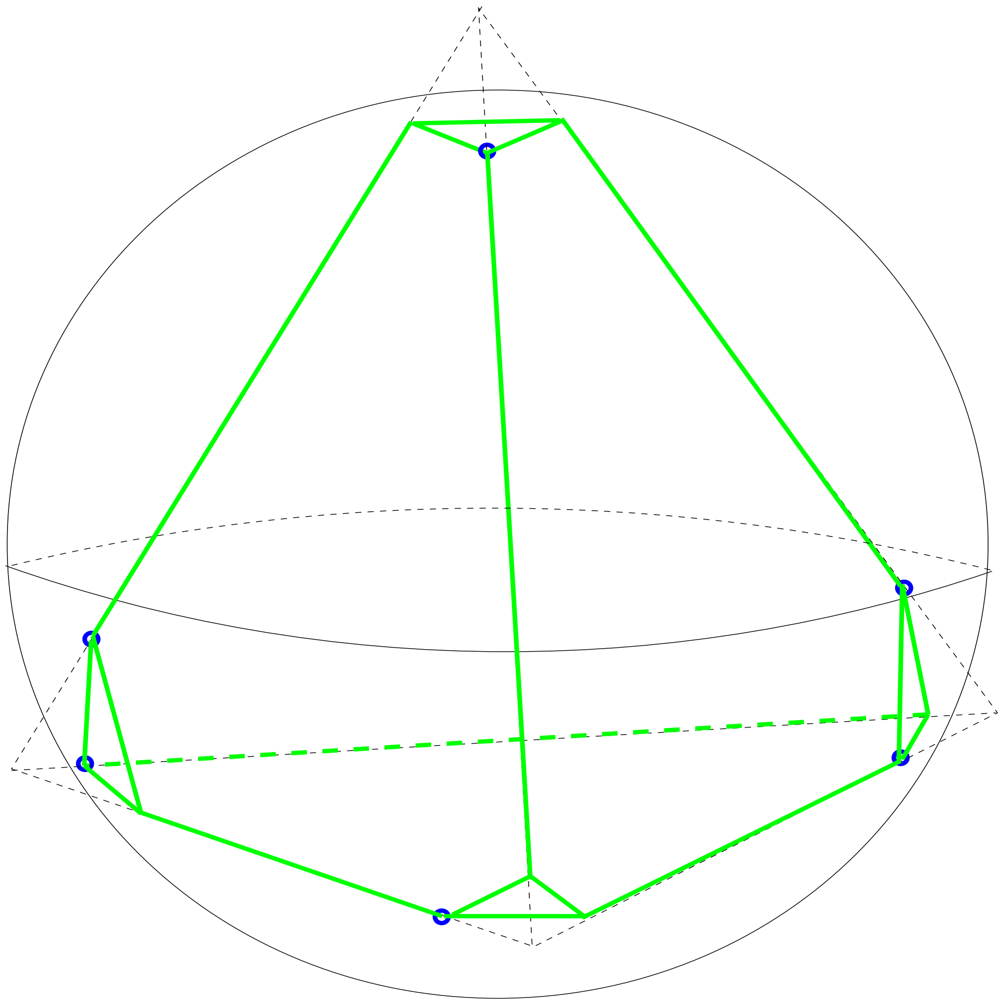
The Convex Supertetrahedron

Let us ignore the polar planes for a moment.



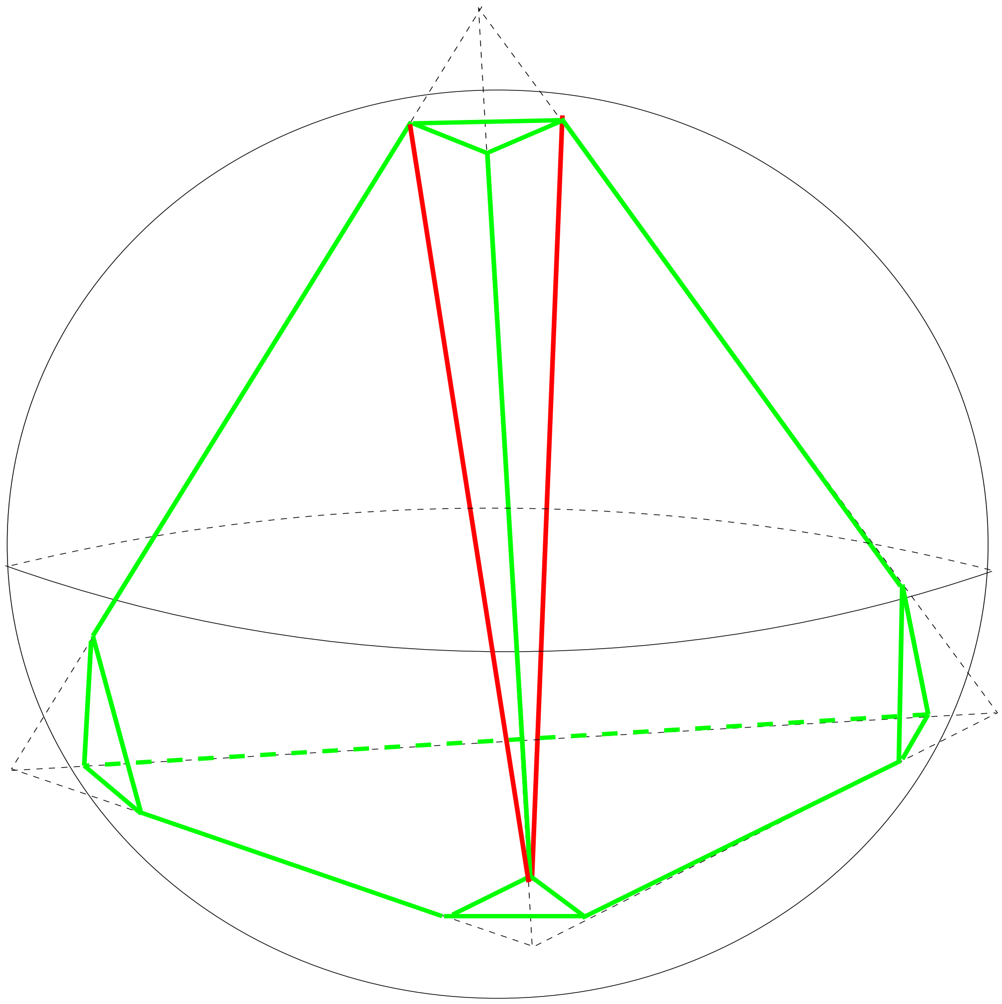
Cutting Down The Convex Supertetrahedron

Our goal will be to simplify this configuration. We will "cut off" the labeled blue vertices.



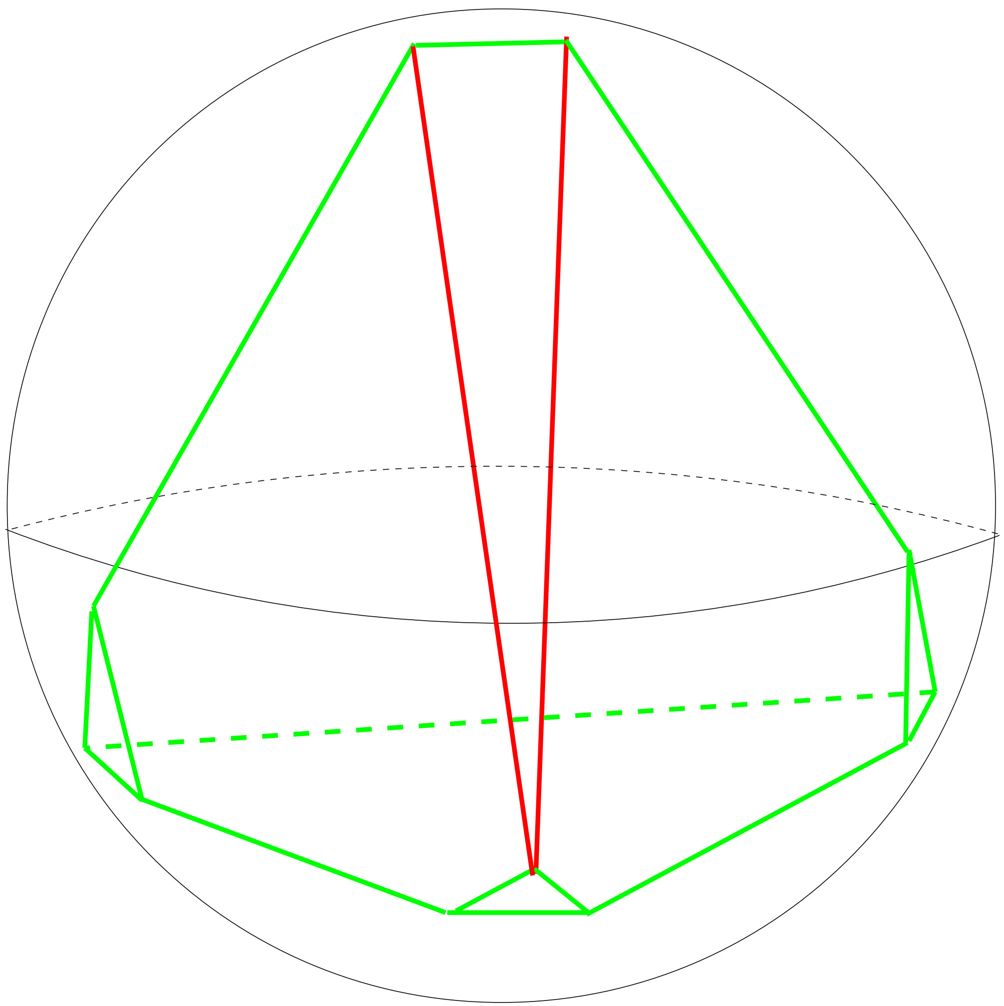
Cutting Down The Convex Supertetrahedron

We start by cutting off this ideal tetrahedron.



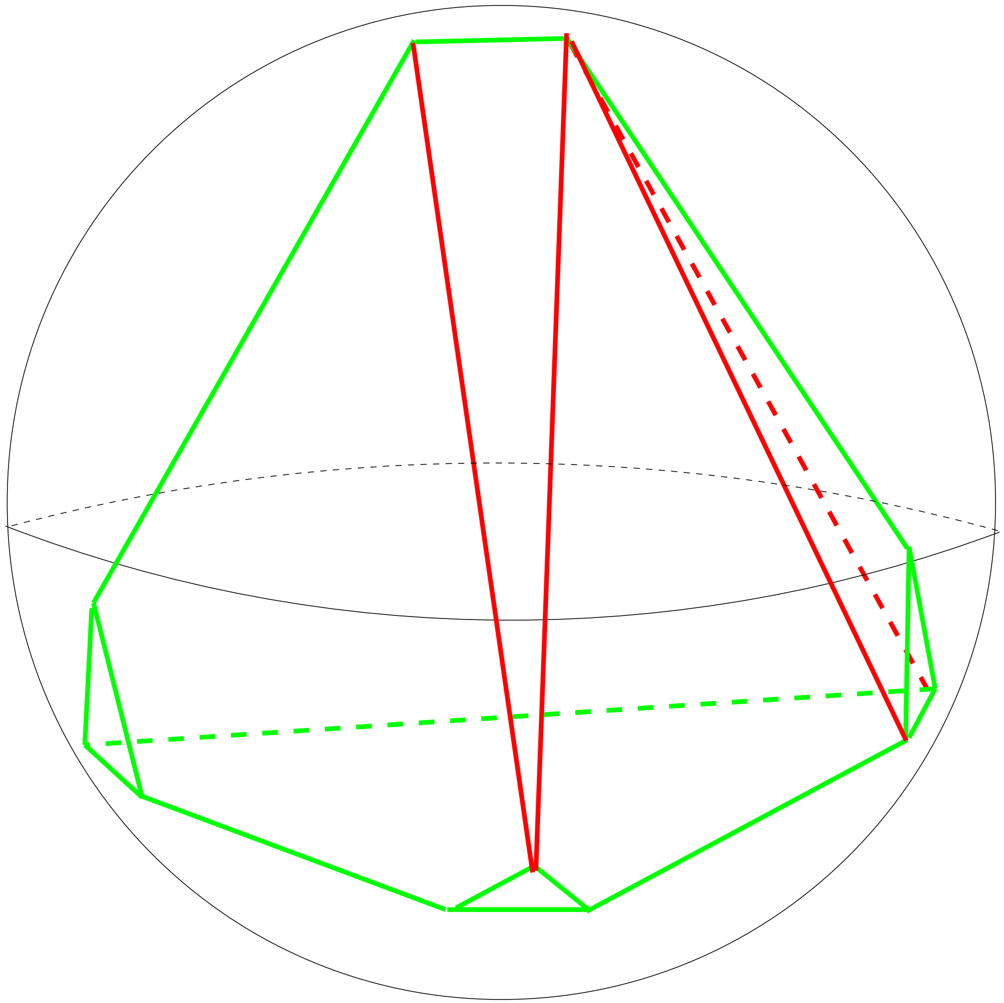
Cutting Down The Convex Supertetrahedron

Recall opposite angles of an ideal tetrahedron are equal, hence we still know the dihedral angles.



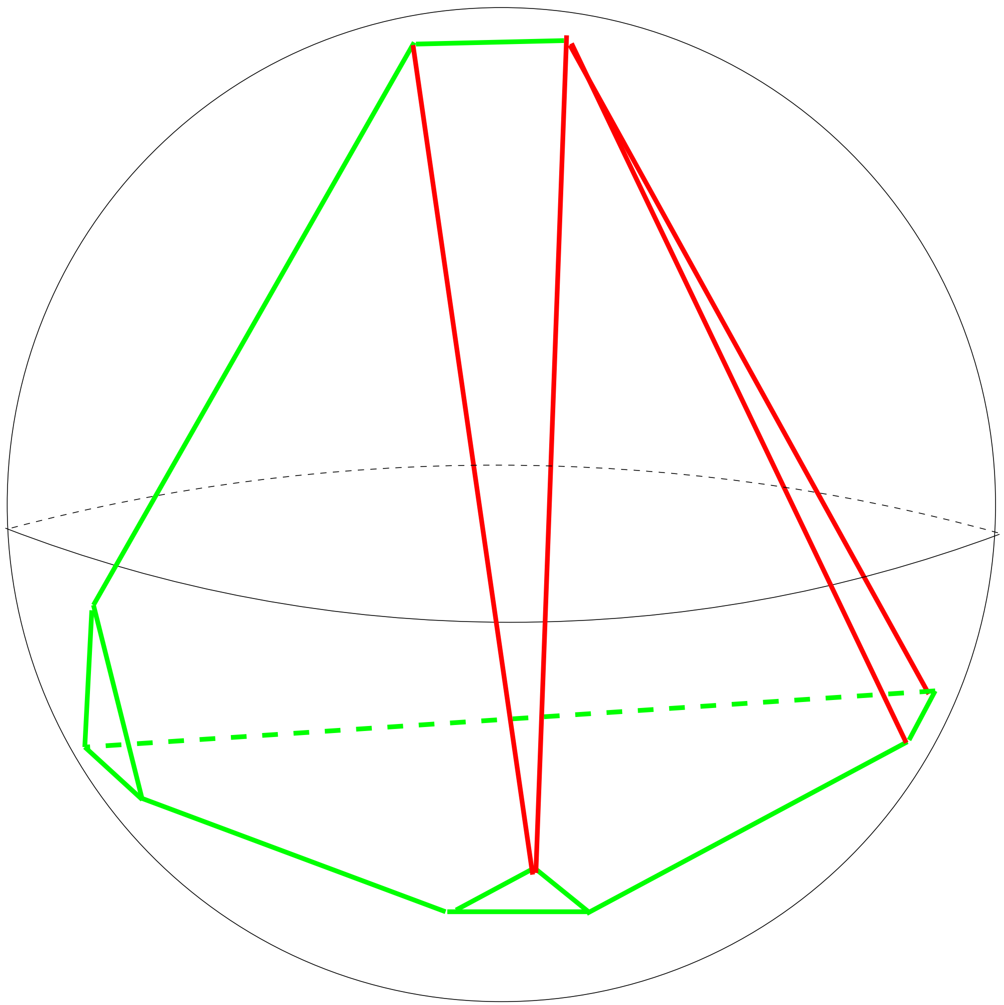
Cutting Down The Convex Supertetrahedron

"Analytic continuation" here refers to the notion that this construction reflects the finite case in important ways.



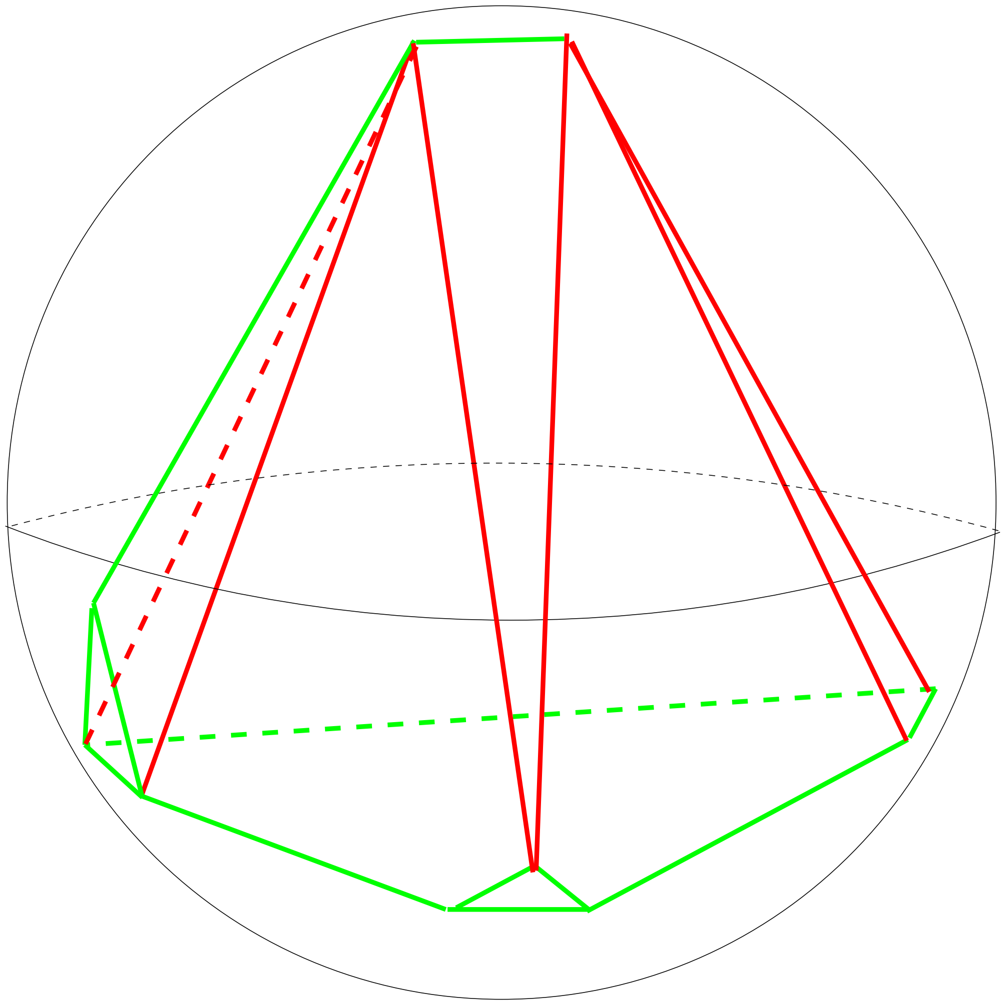
Cutting Down The Convex Supertetrahedron

Most importantly, the relationships between the clinants are preserved.



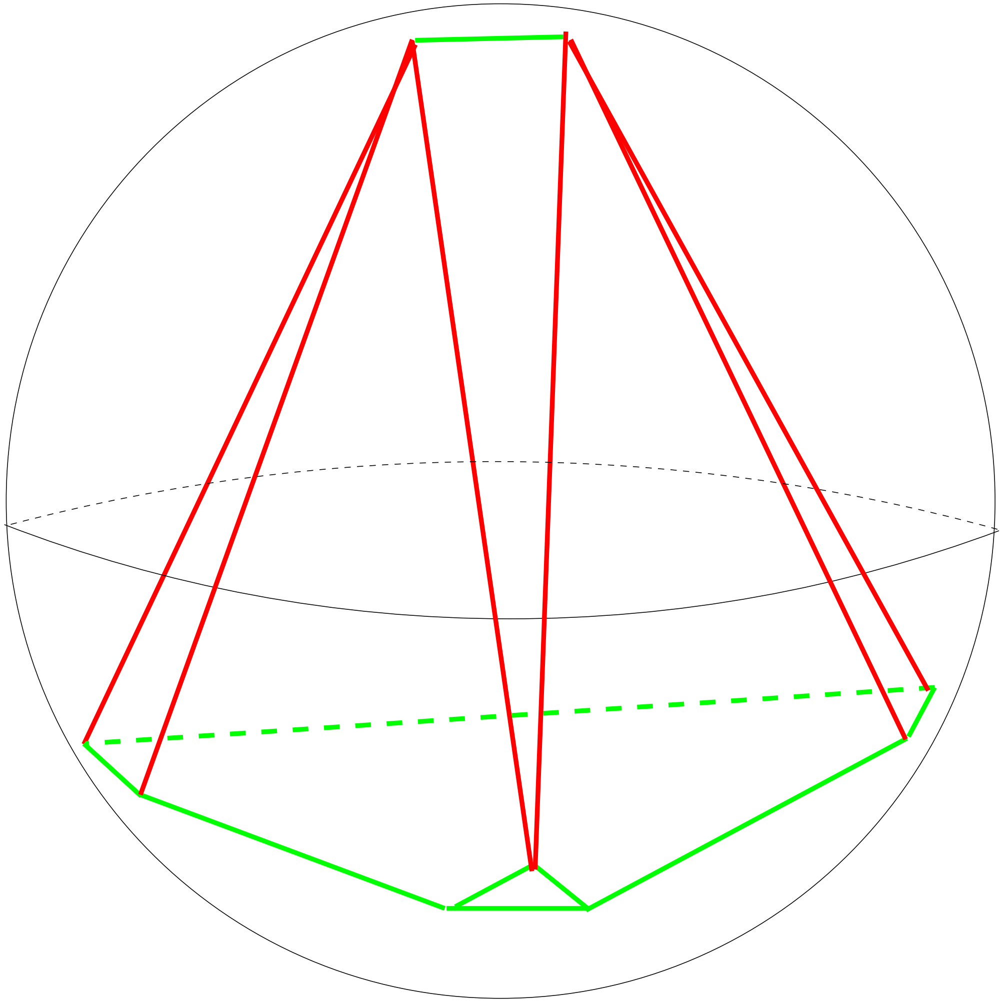
Cutting Down The Convex Supertetrahedron

Recall, the clinants are $e^{2I\theta}$ where θ is a dihedral angle.



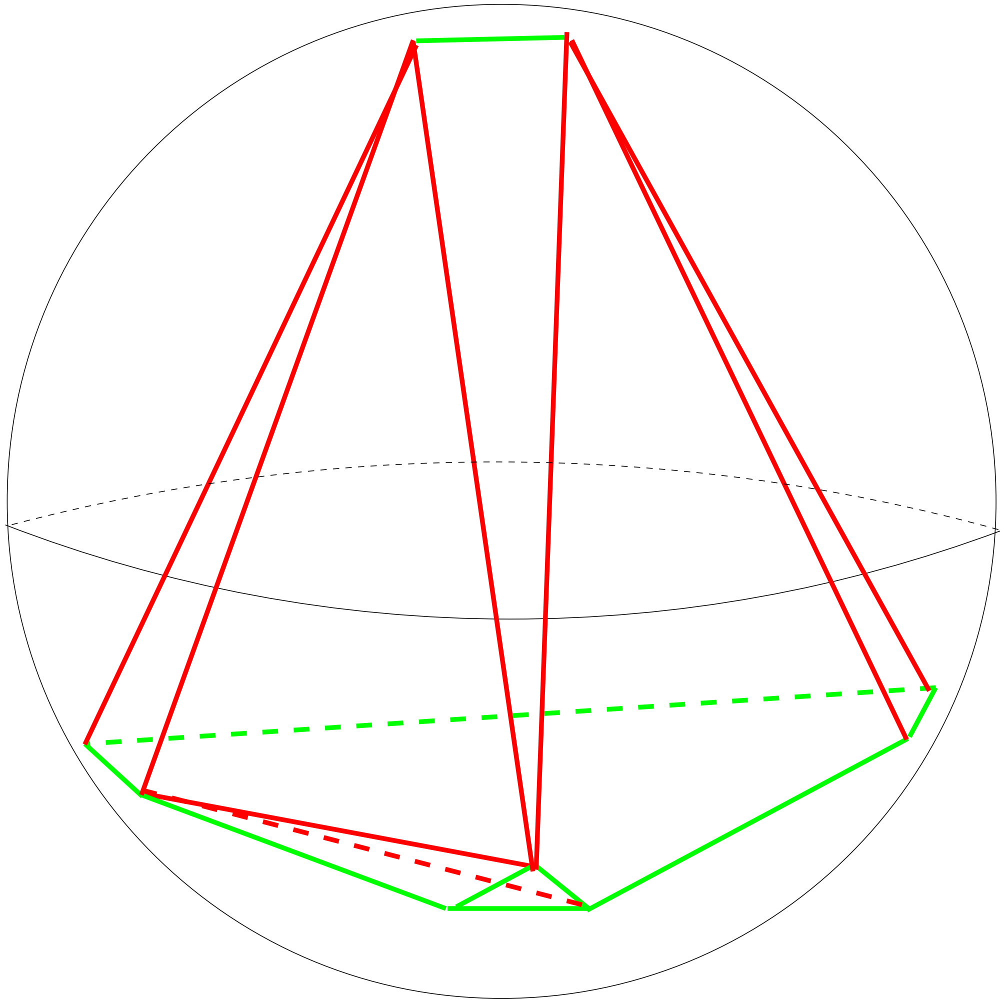
Cutting Down The Convex Supertetrahedron

KEY: the dihedral angles are not preserved under analytic continuation, but the clinants are.



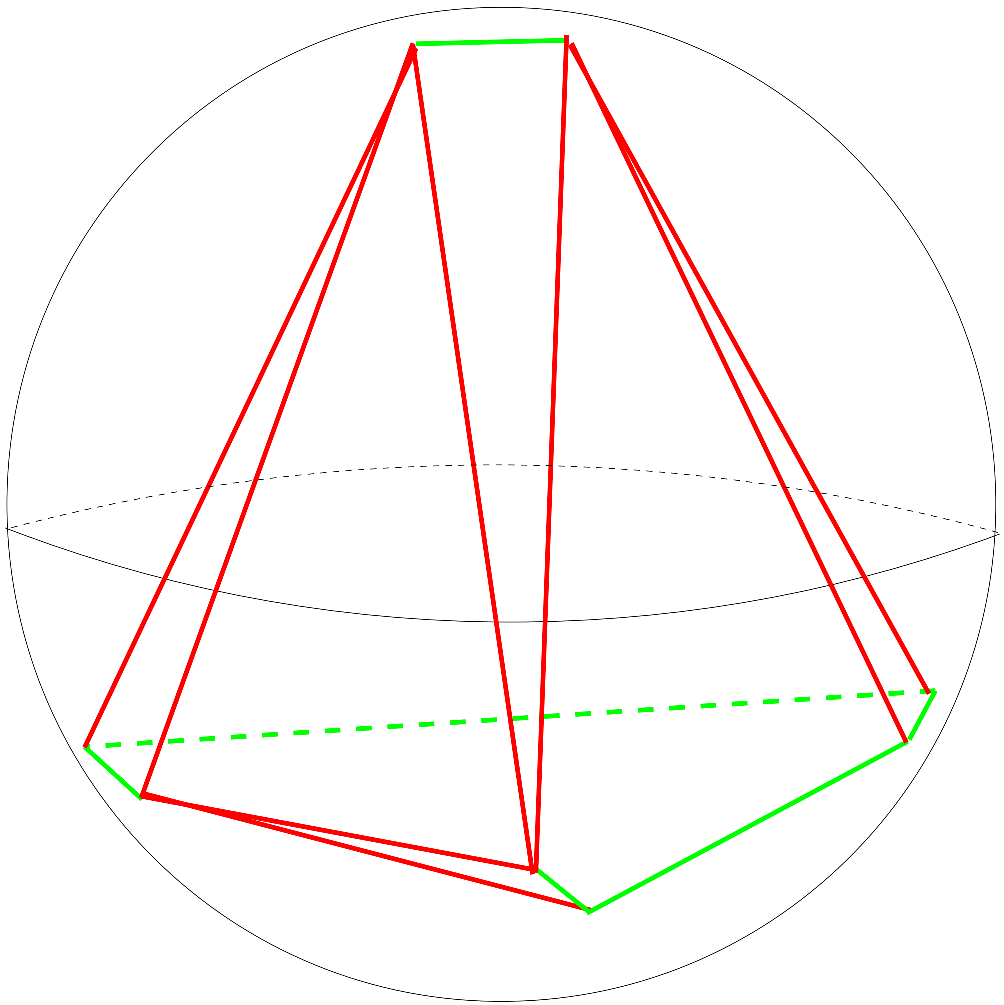
Cutting Down The Convex Supertetrahedron

We have seen that the clinants of an ideal tetrahedron determine it up to isometry.



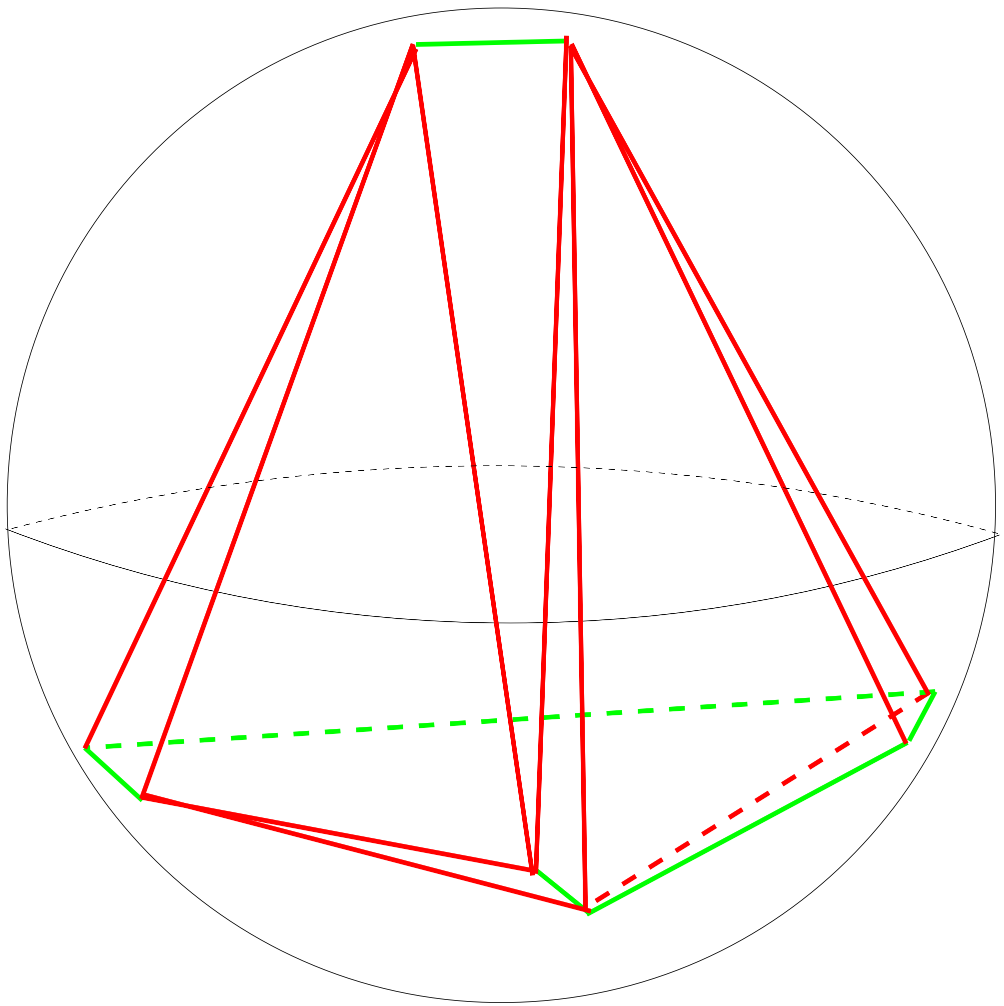
Cutting Down The Convex Supertetrahedron

In particular, any decomposition into ideal tetrahedra is analytically continuable.



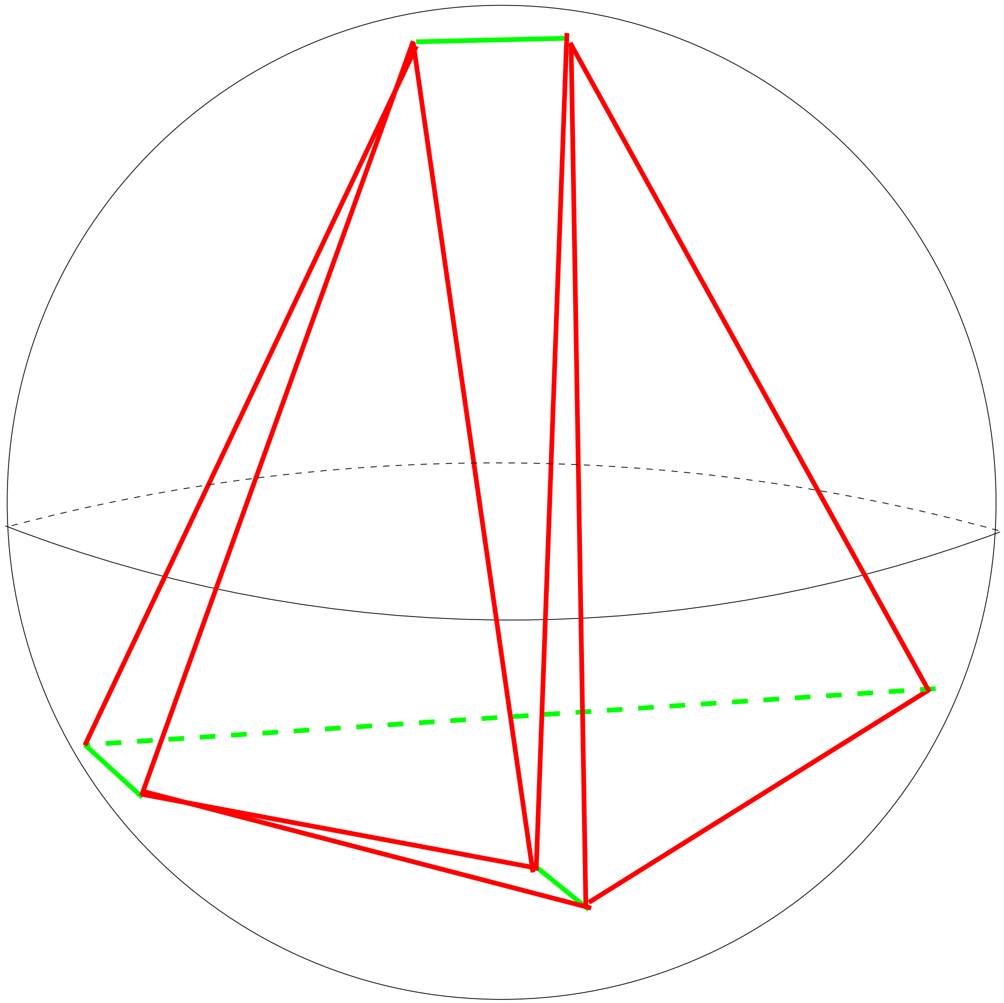
Cutting Down The Convex Supertetrahedron

Hence, any volume formula describe via an ideal decomposition can be analytically continued.



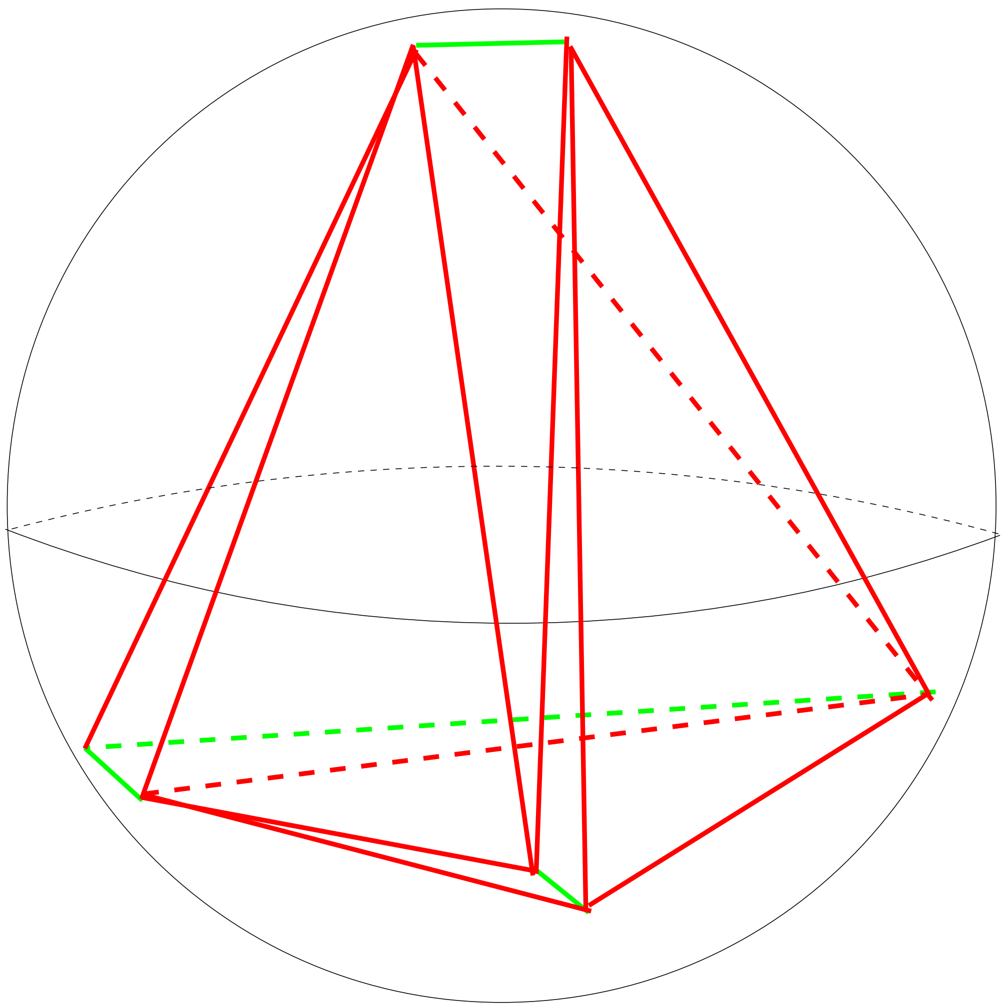
Cutting Down The Convex Supertetrahedron

As a result, finding a formula for volume of the supertetrahedron has been reduced to finding a scissors class representation of the convex supertetrahedron in terms of ideal tetrahedra.



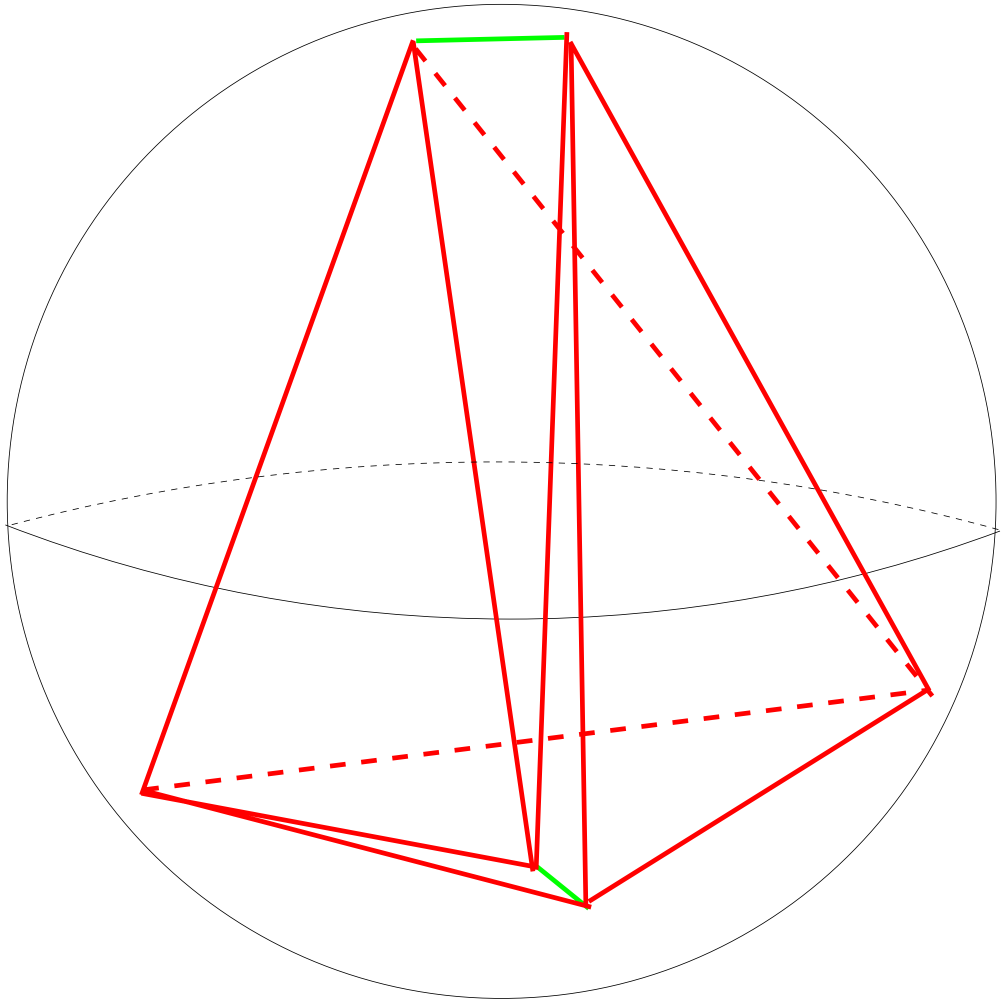
Cutting Down The Convex Supertetrahedron

The ideal tetrahedron in our pictured cut down will contribute to one possible formula.



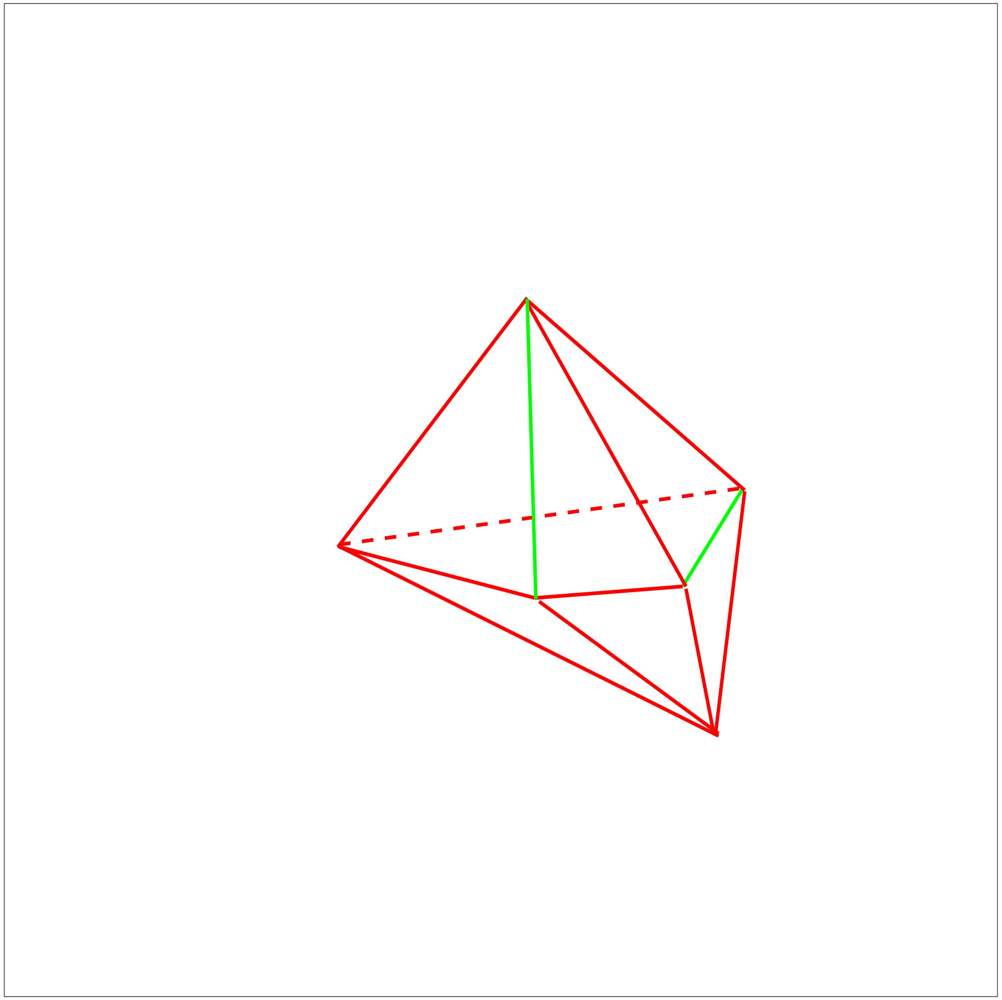
Cutting Down The Convex Supertetrahedron

If we cut till we can cut no more, then we always cut down to an octahedron.



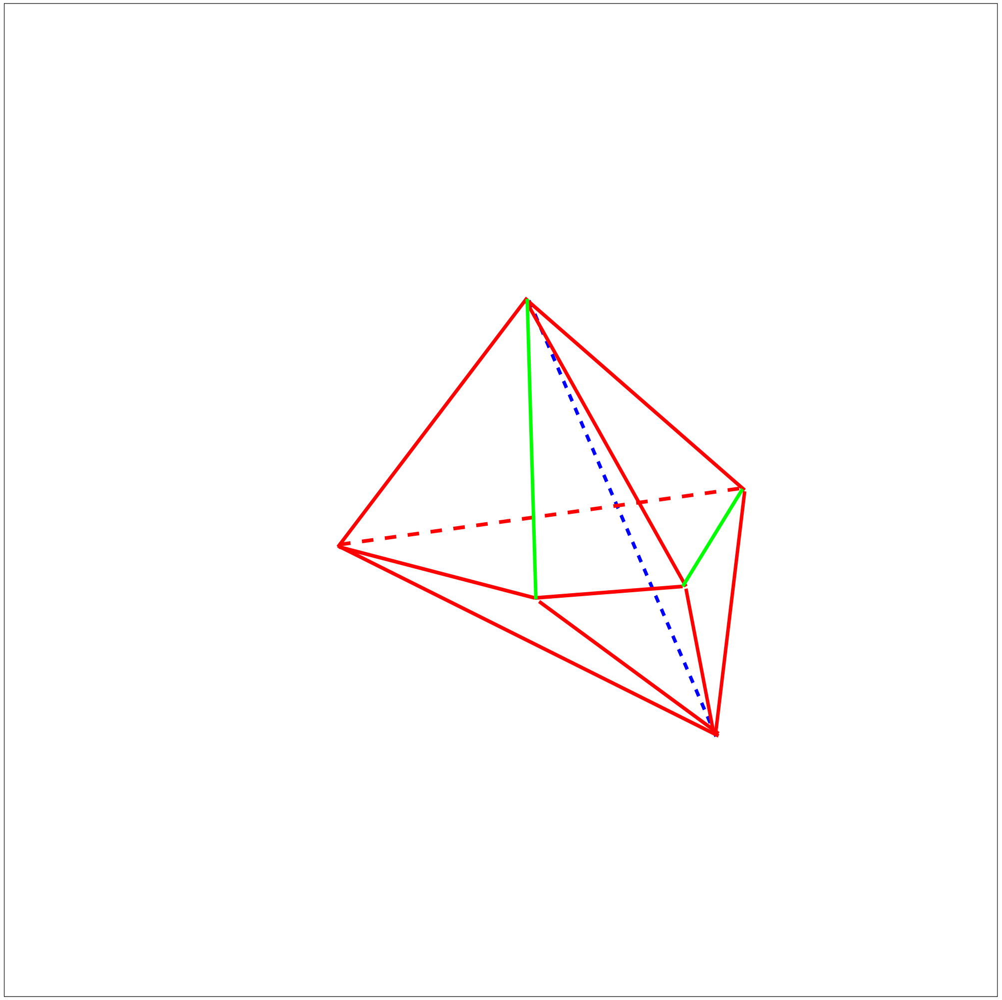
Cutting Down The Convex Supertetrahedron

Let us pick a nicer view of this octahedron...



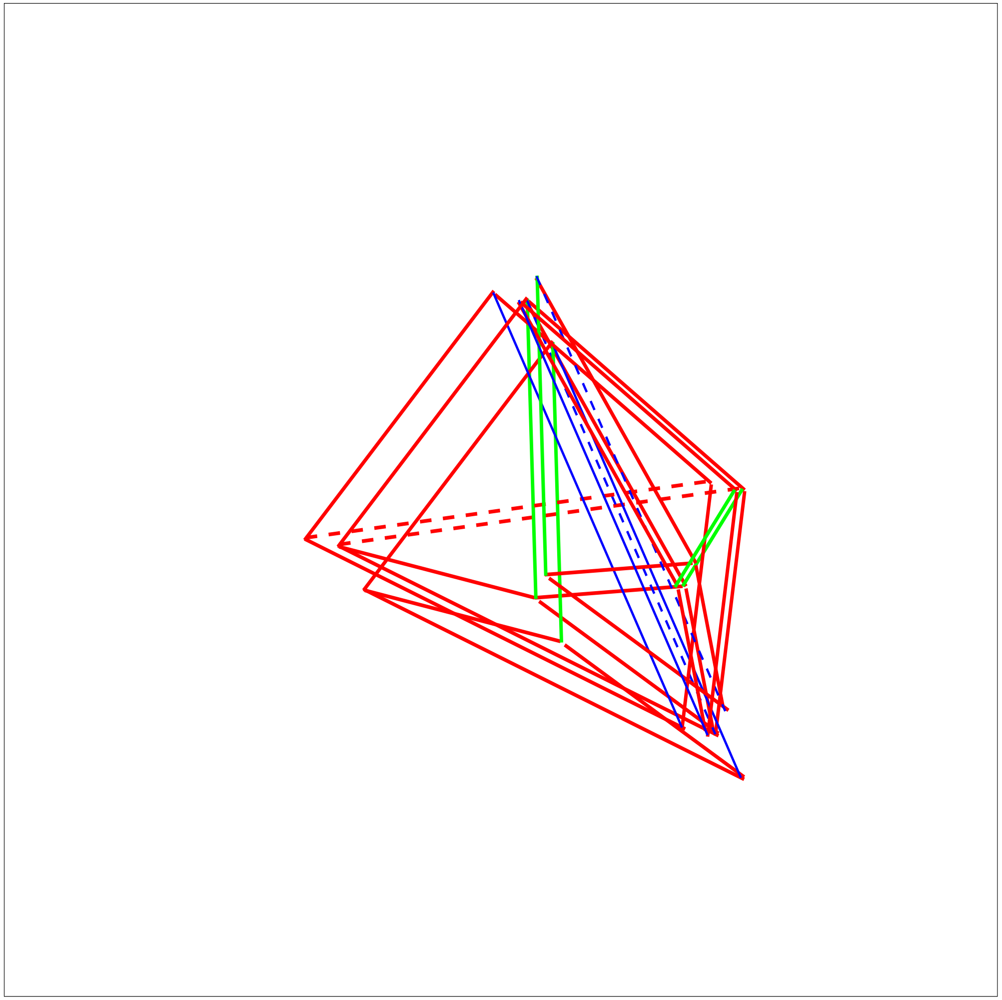
Cutting Down The Convex Supertetrahedron

and a blue "fire pole".



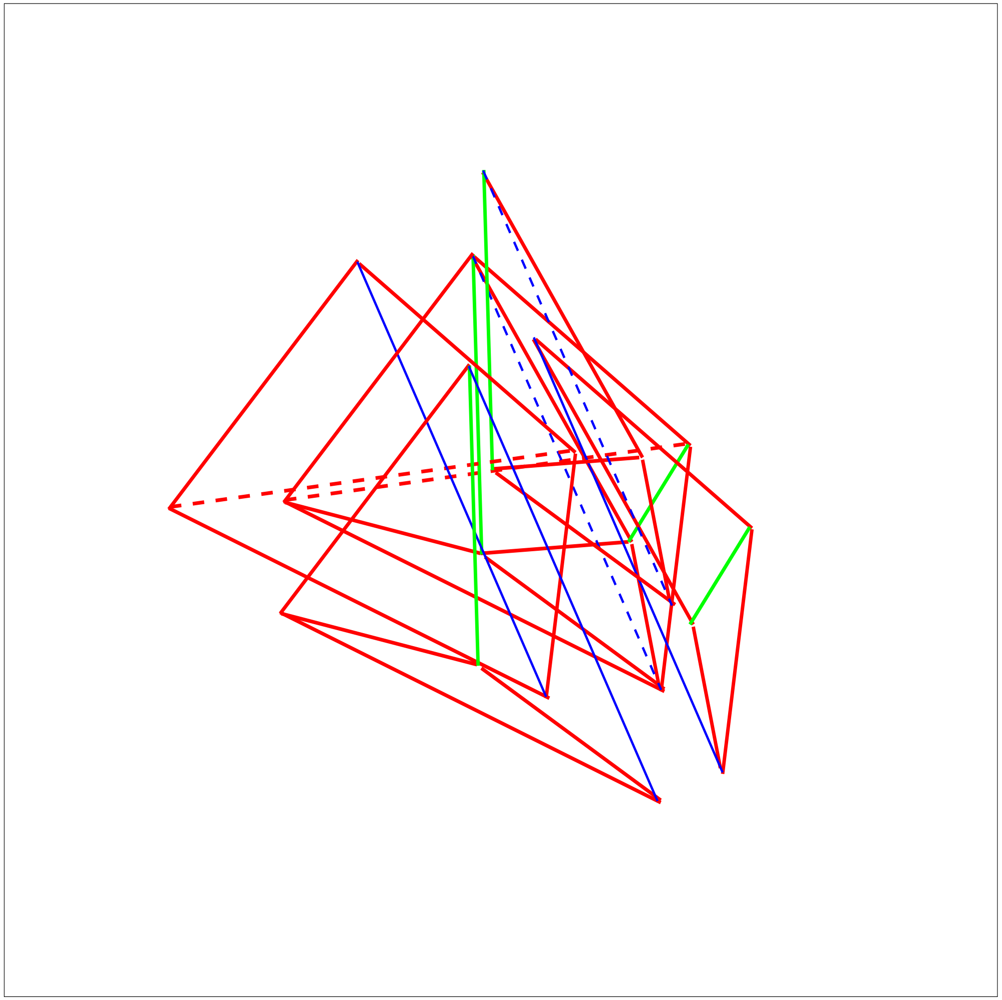
Cutting Down The Convex Supertetrahedron

Notice, we can split along this fire pole...



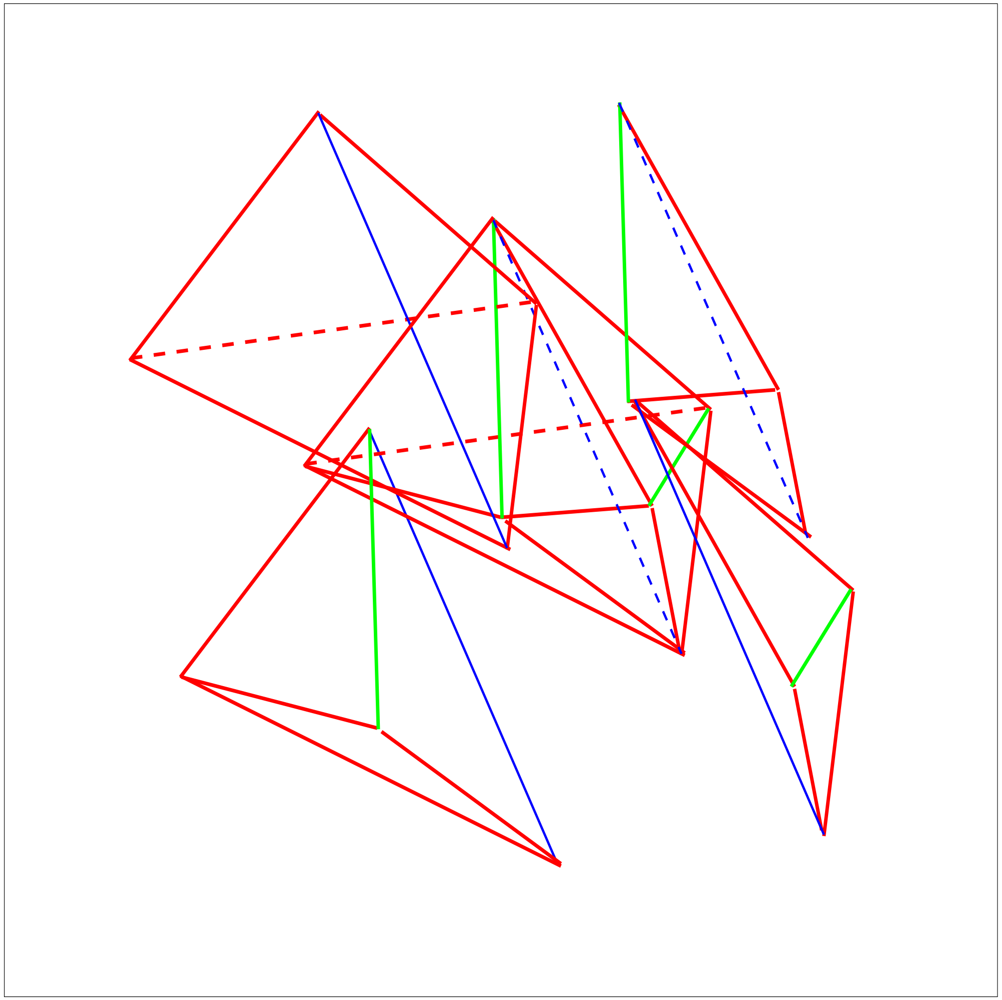
Cutting Down The Convex Supertetrahedron

and...



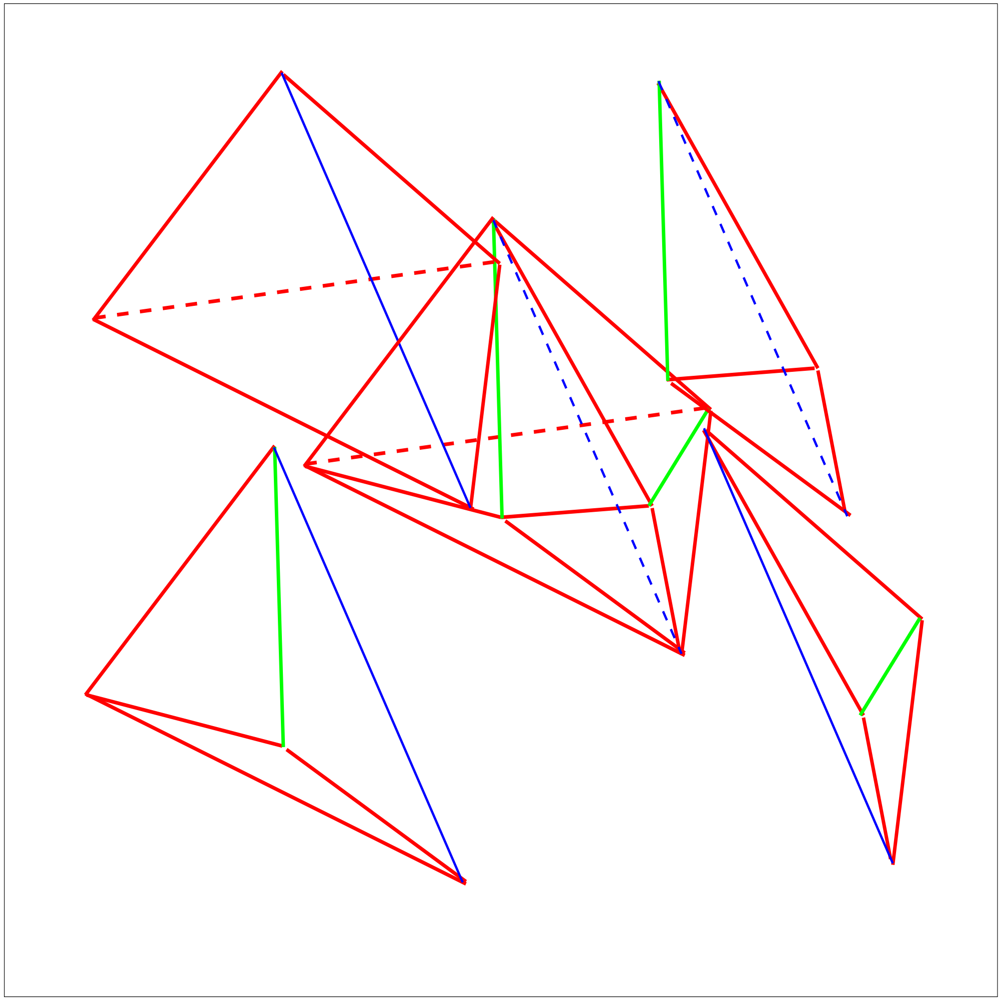
Cutting Down The Convex Supertetrahedron

and...



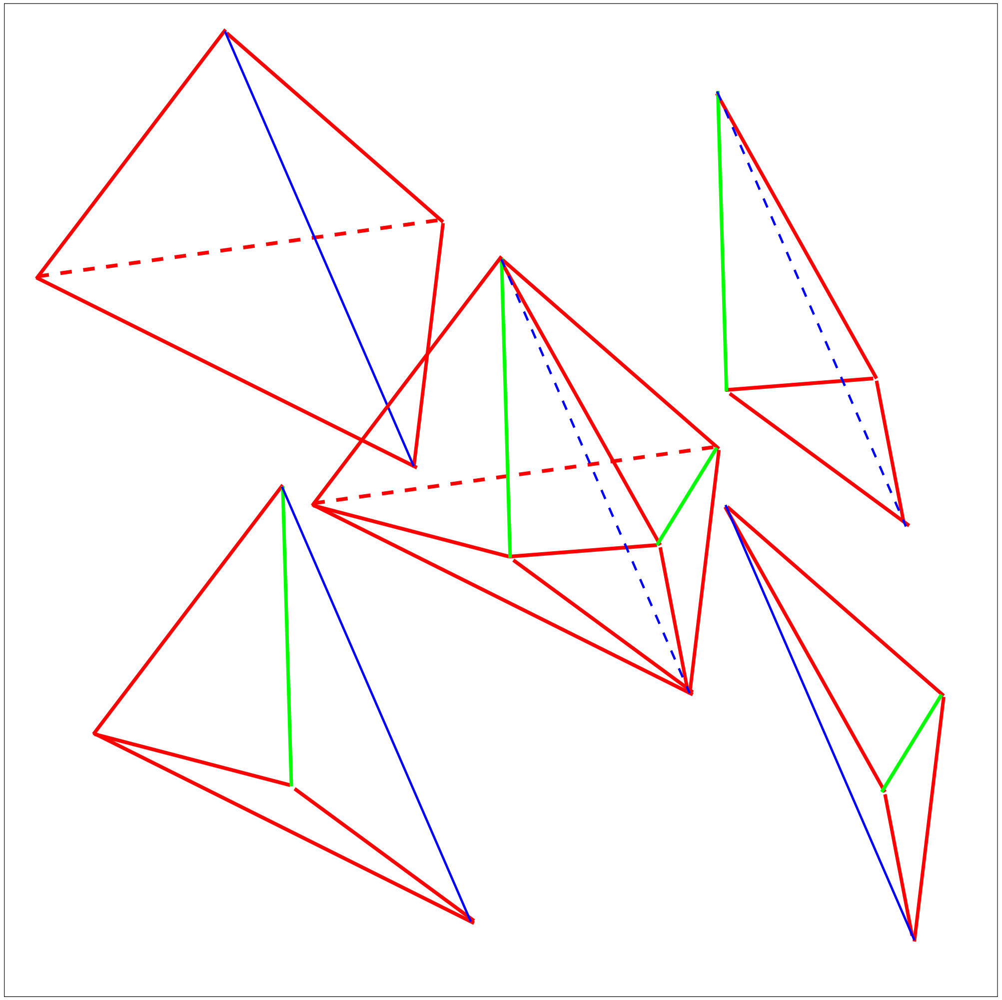
Cutting Down The Convex Supertetrahedron

and...



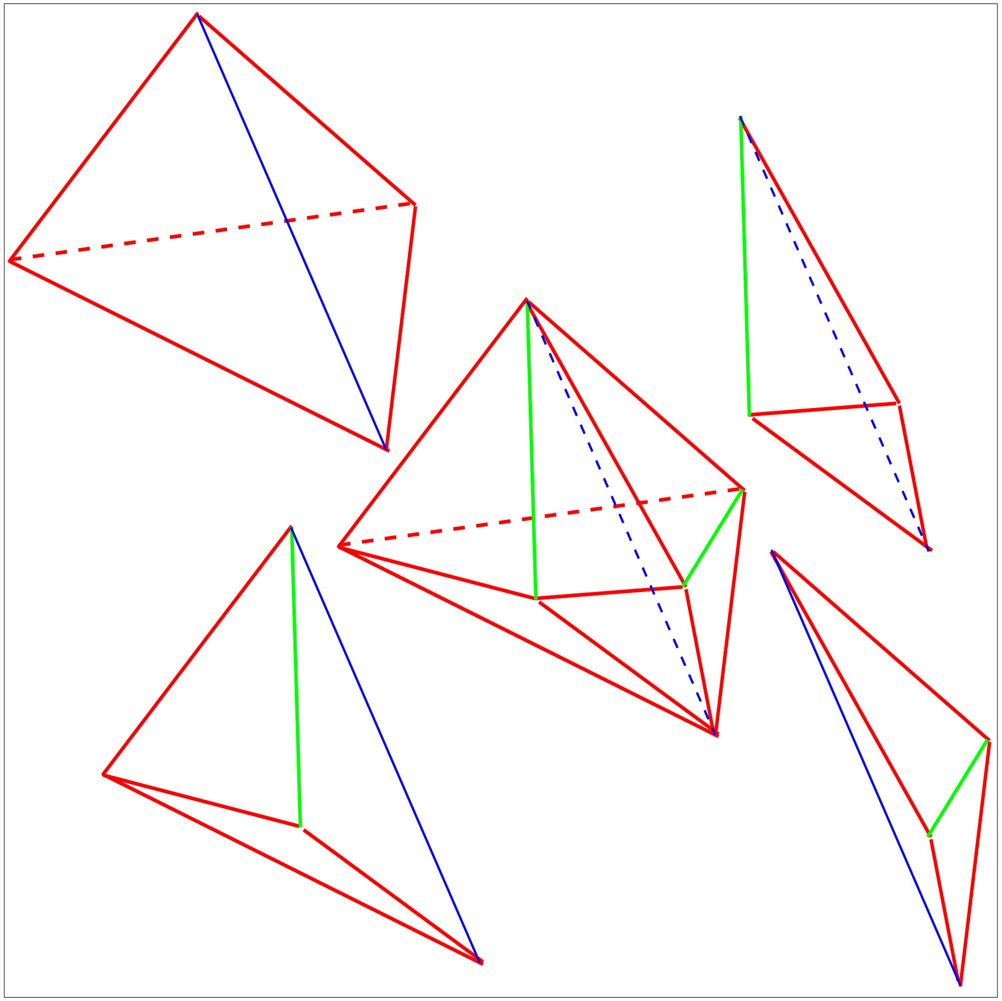
Cutting Down The Convex Supertetrahedron

decompose our octahedron into four ideal tetrahedra..



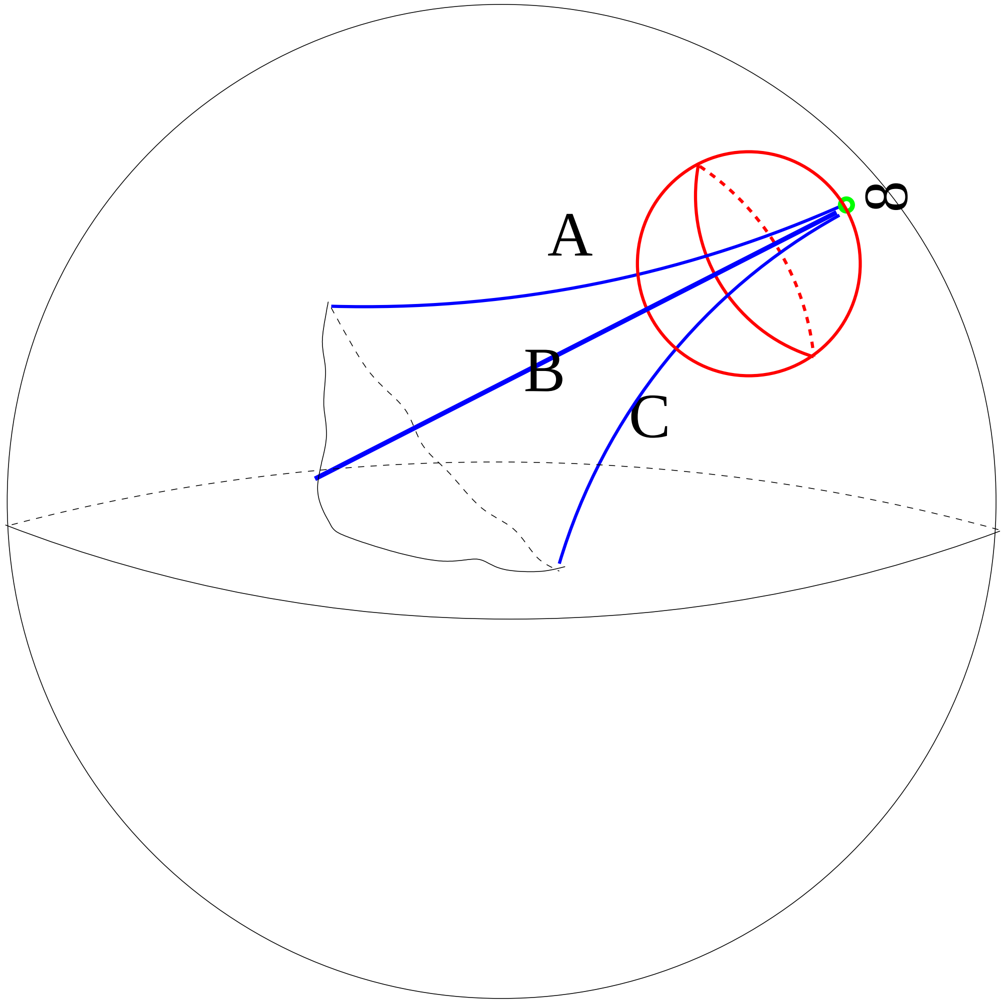
Cutting Down The Convex Supertetrahedron

Hence, finding the needed volume formula has been reduced to understanding the ideal tetrahedra used to form a convex octahedron.



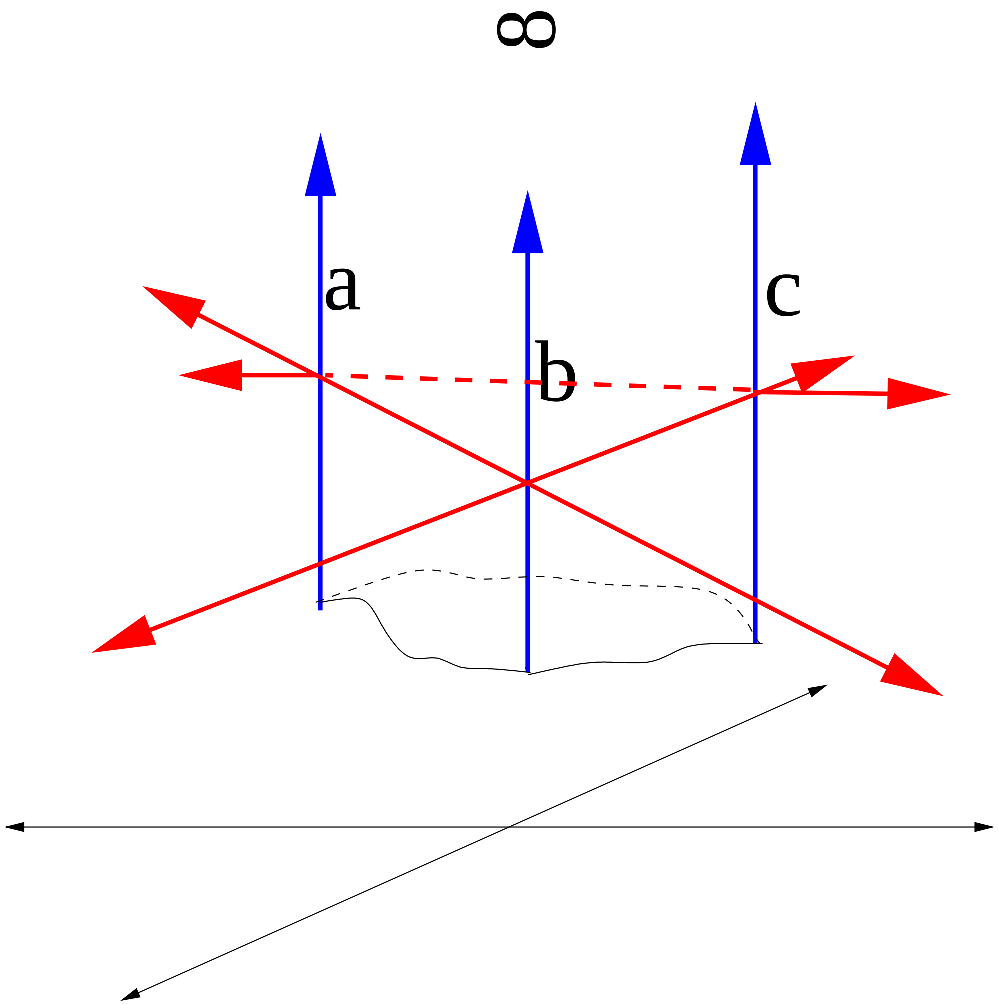
Euclidean Angles

To accomplish this, we must carefully keep track of all the clinants. Recall at each ideal vertex of an ideal polyhedron we see this picture, where the red sphere is a horosphere.



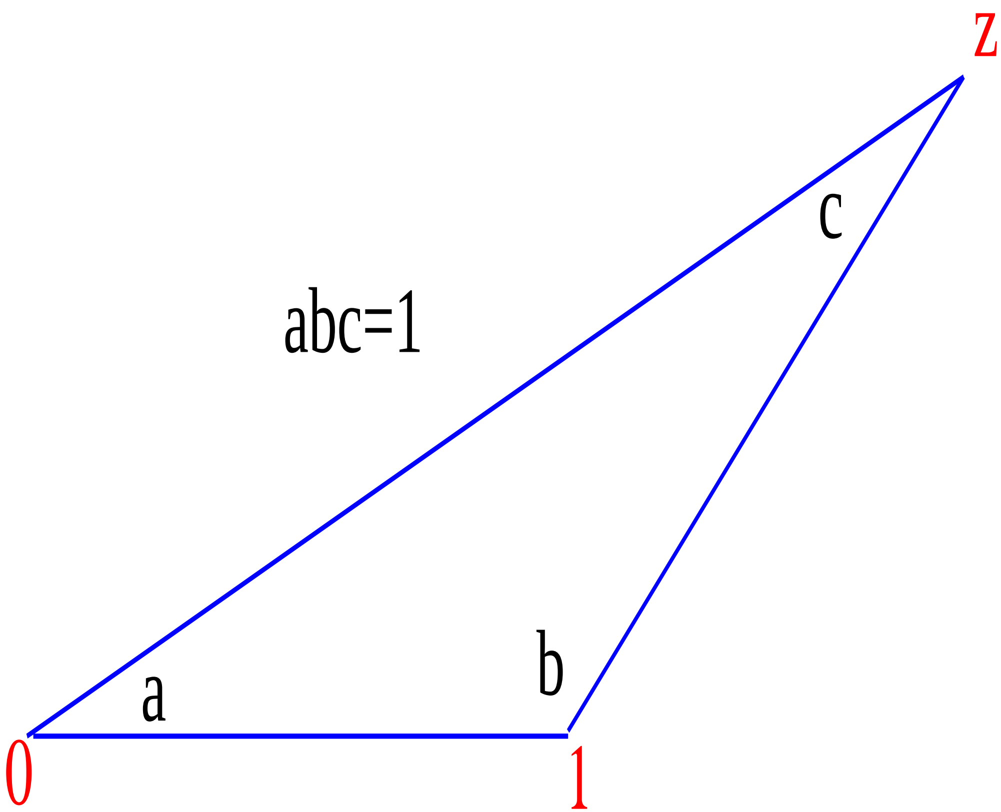
Euclidean Angles

Sending the ideal vertex to the point at infinity in the upper-half space model, we find that the clinants at an ideal vertex are Euclidean and multiply to 1.



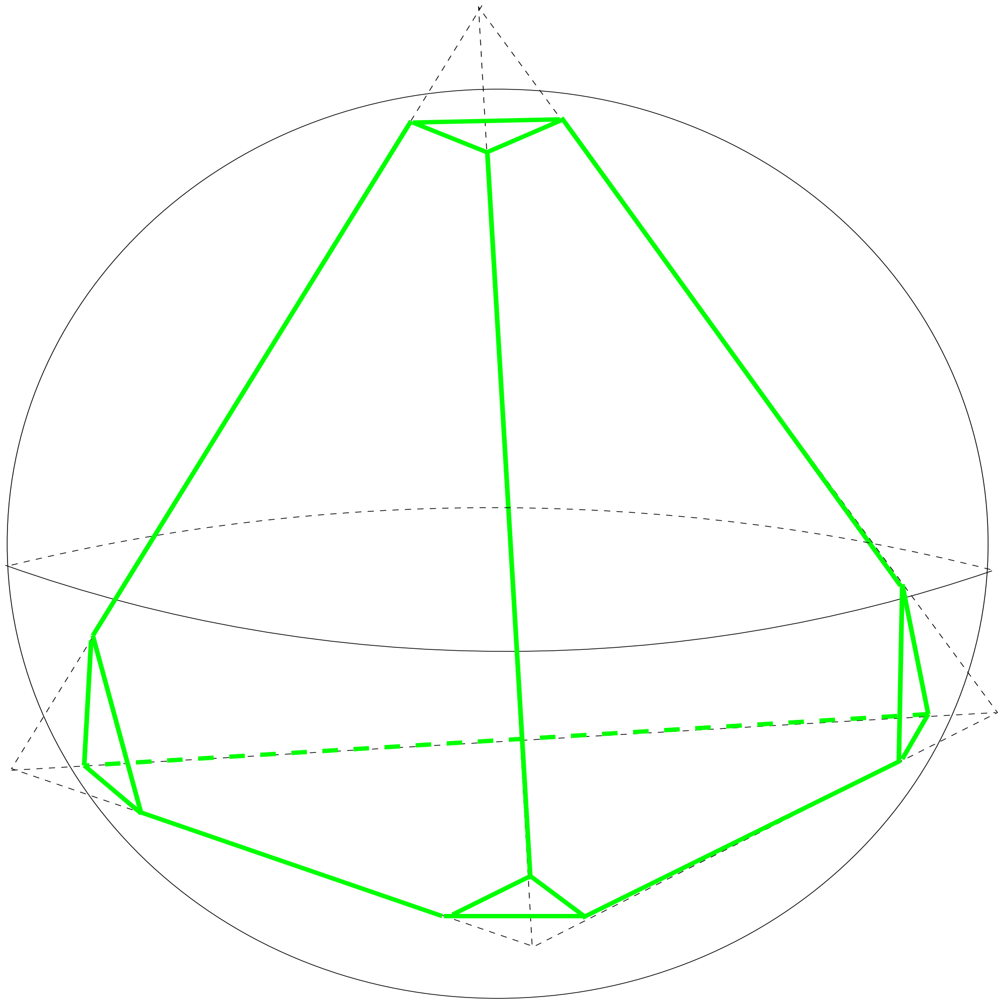
The Ideal Tetrahedron's Clinants

Recall the space of ideal tetrahedra is parameterized by clinant triples (a, b, c) such that $abc = 1$ ("blown up" at $(1, 1, 1)$). Also recall, the z coordinate is easily determined by the clinants and equals $\frac{1-a}{1-\bar{b}}$.



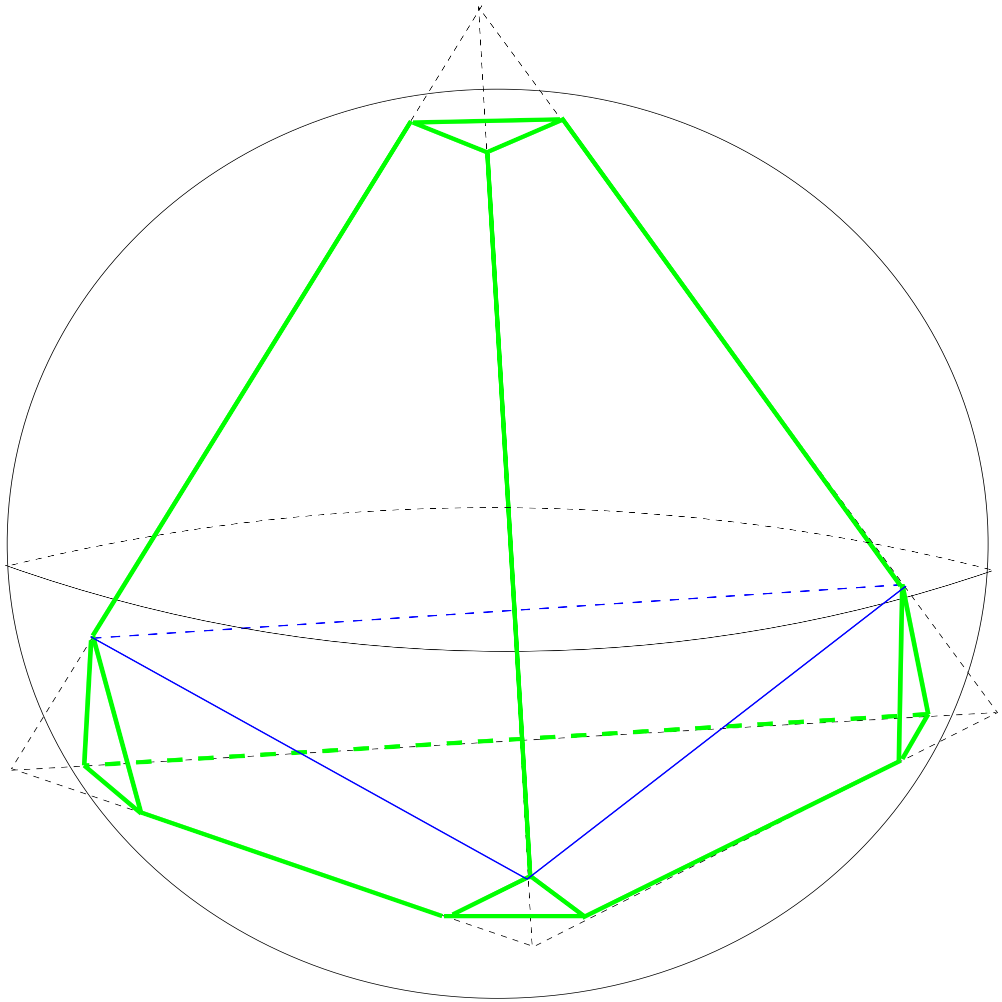
The Supertetrahedron's Angles

Notice all our supertetrahedron's original angles are tetrahedral angles or...



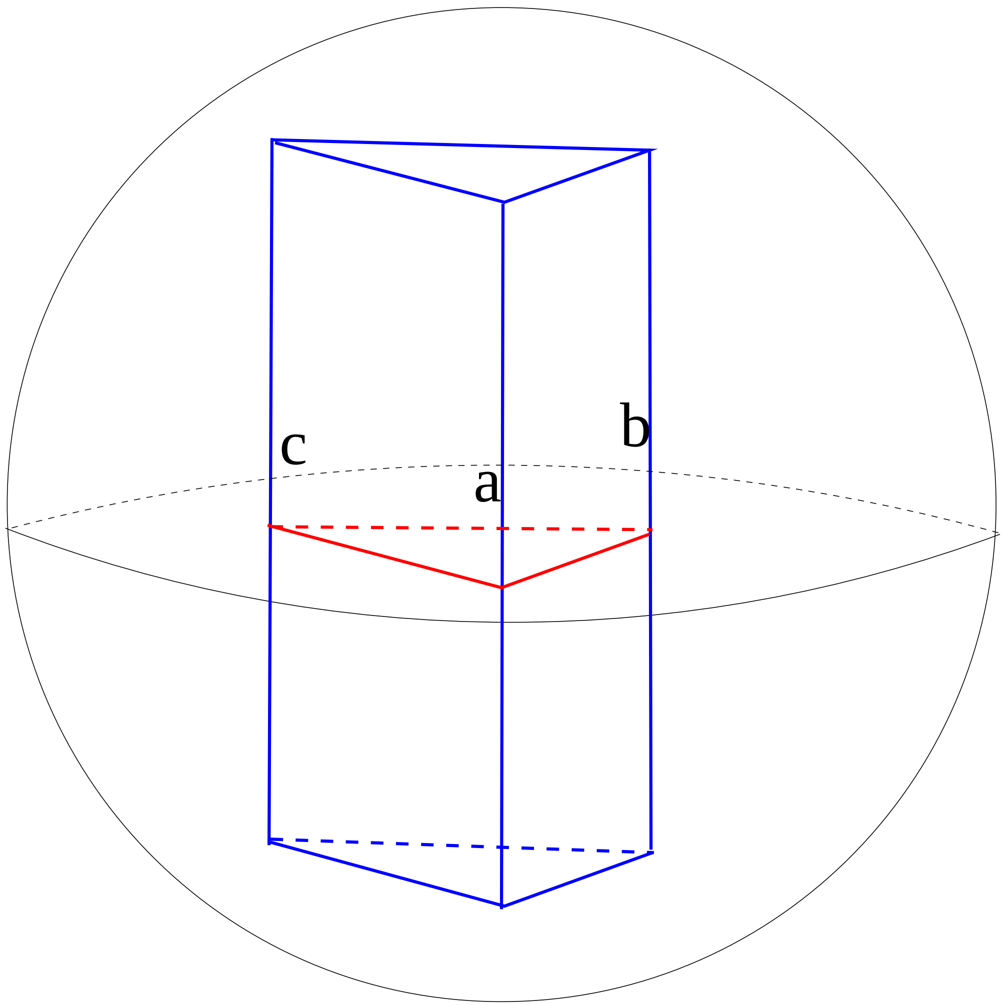
The Supertetrahedron's Angles

...prism angles.



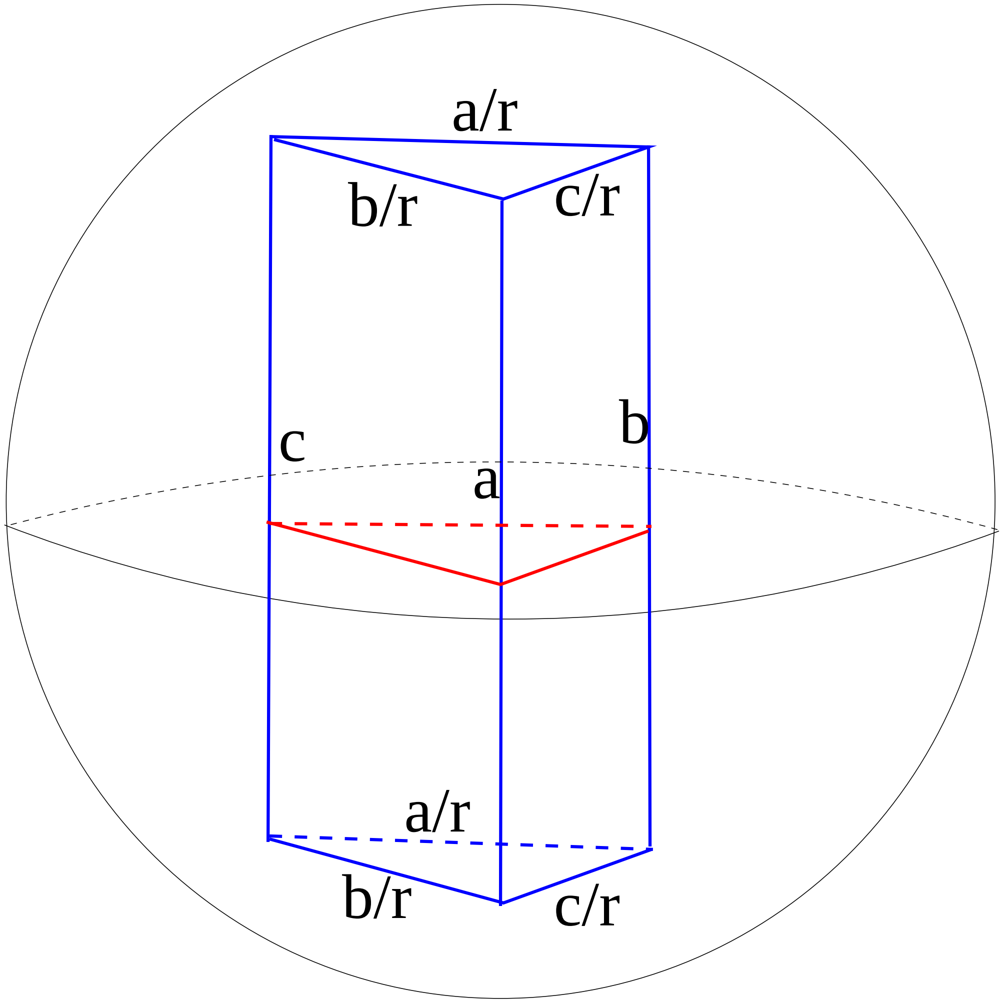
The Prism Angles

Hence, we need to determine the prism’s clinants.



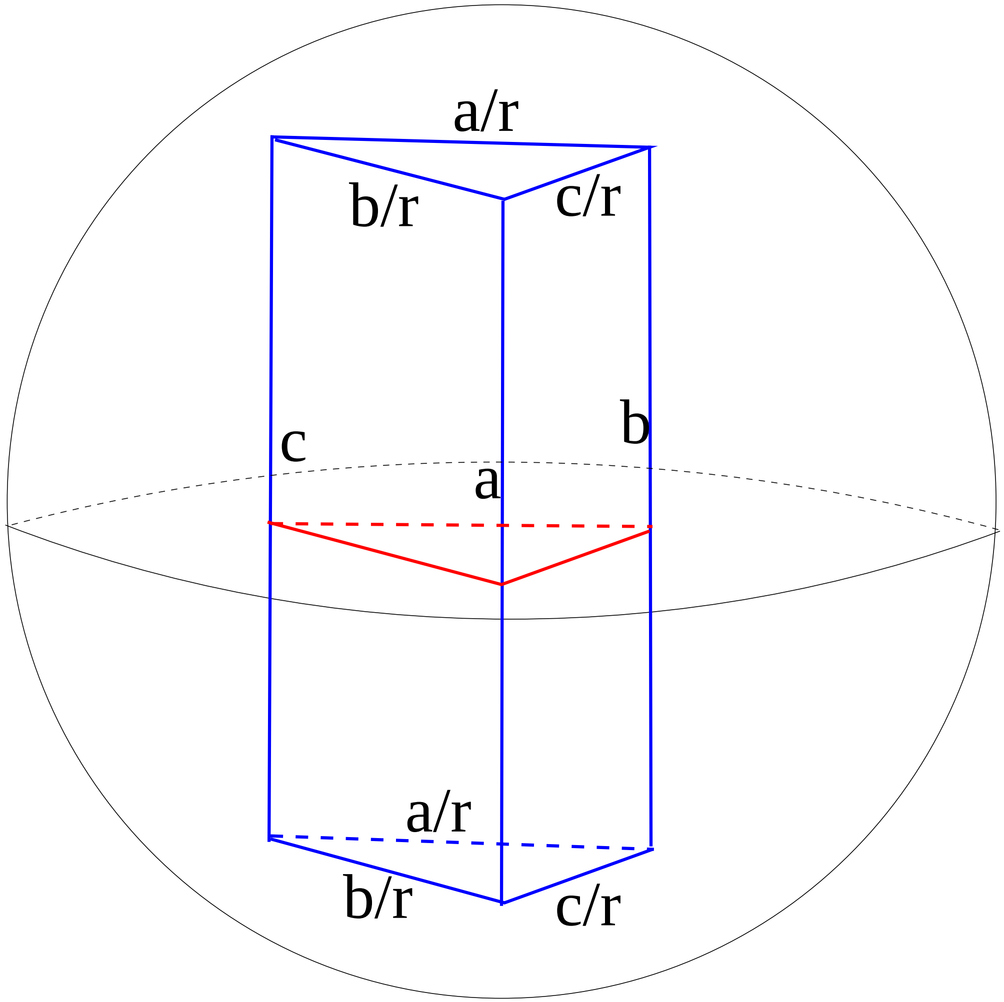
The Prism Root

Letting $r = -e^{I(A+B+C)}$ and using the fact that the ideal vertices are Euclidean, we can determine the following clinants:



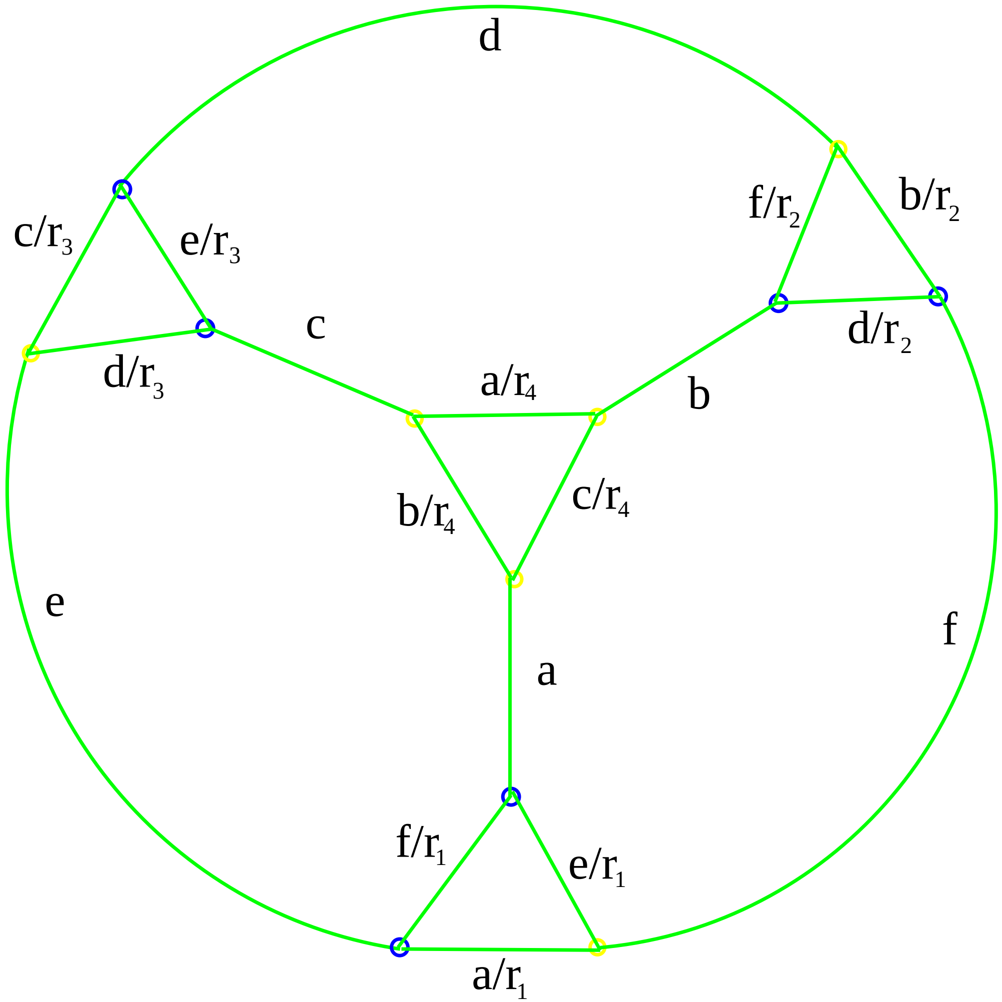
The Decomposition's Angles

Now we will once again cut down our supertetrahedron. This time we will keep track of the clinants. Keep in mind that the opposite clinants of an ideal tetrahedron are equal.



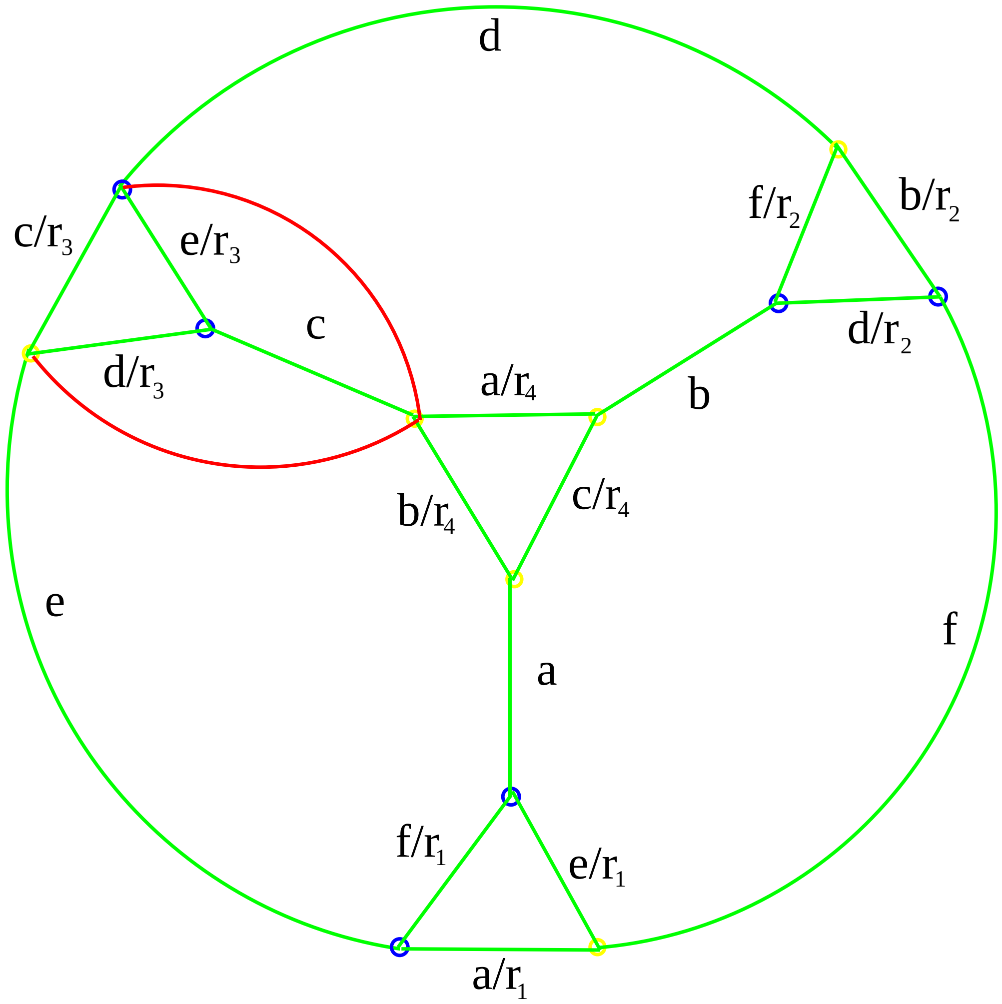
The Decomposition's Angles

We use a combinatorial view of a supertetrahedron.



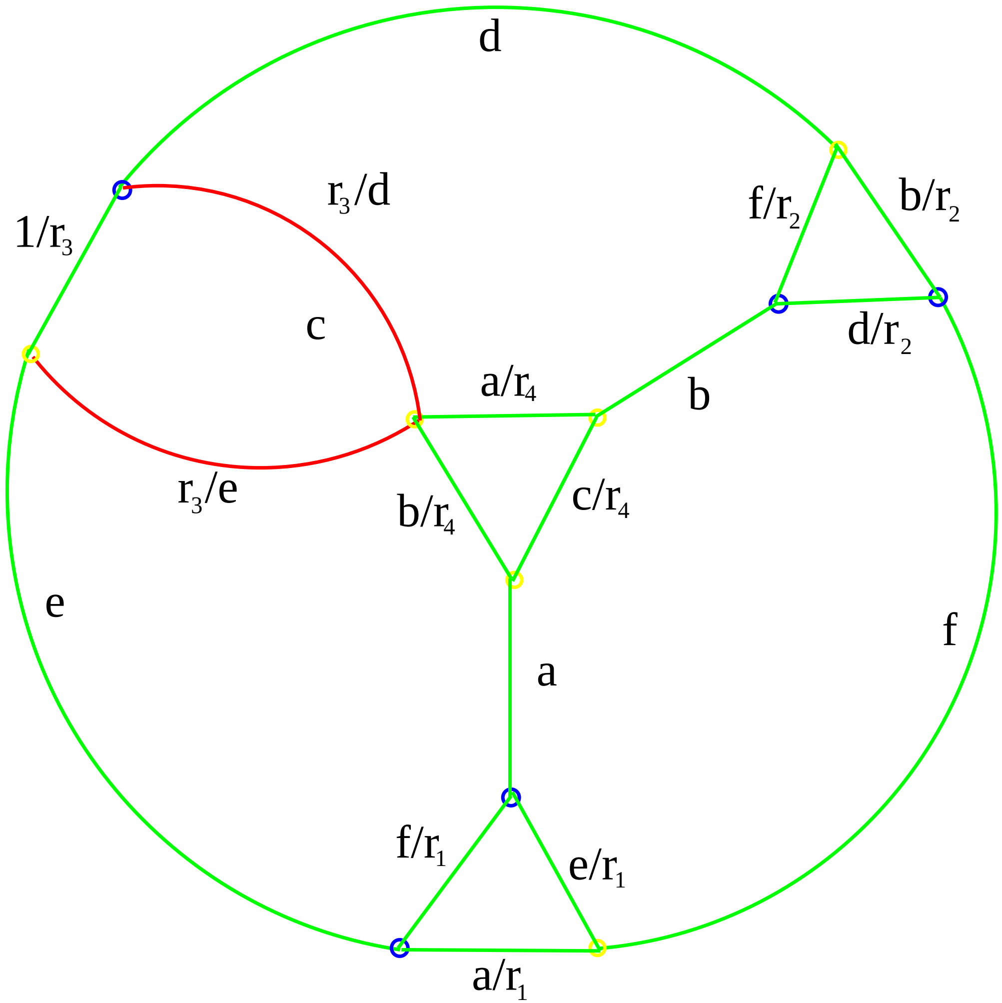
Another Cut Down

Once again let us cut off the blue vertices.



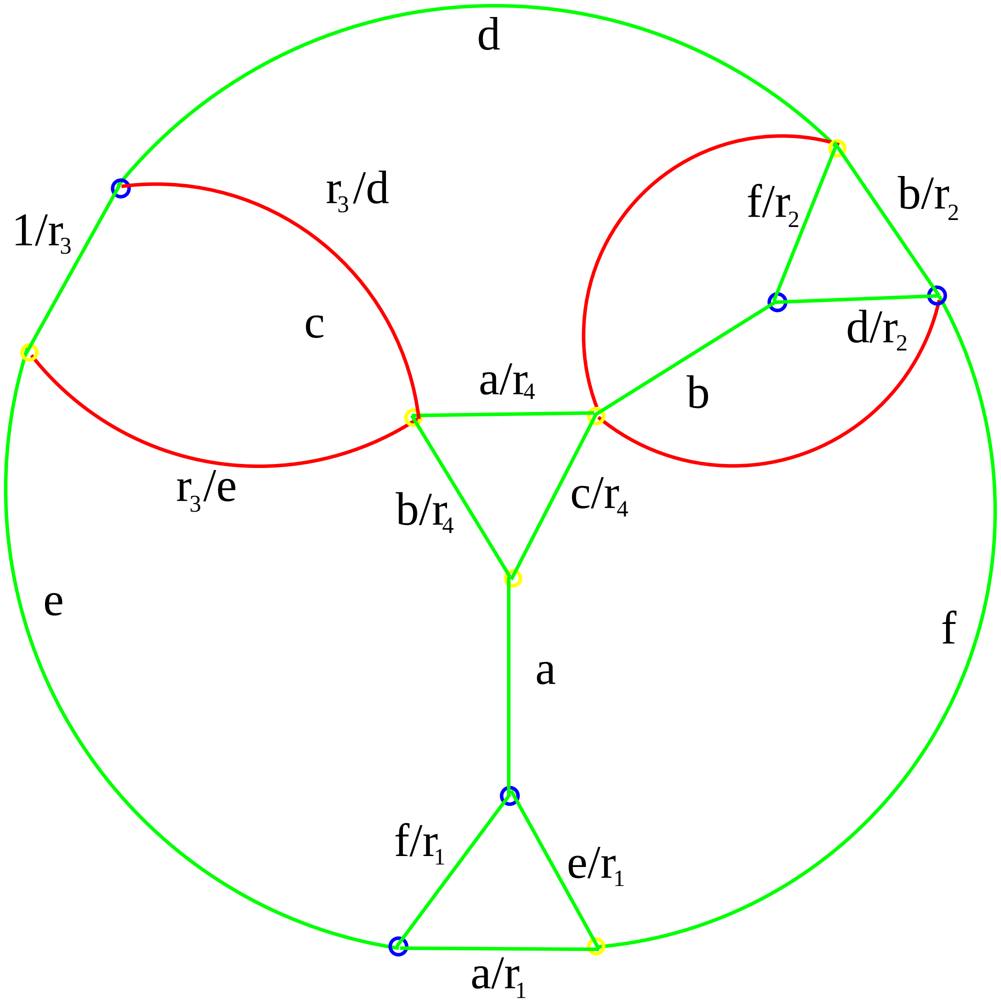
Another Cut Down

Since opposite angles in an ideal tetrahedron are equal, and we hence we know the clinants after a cut.



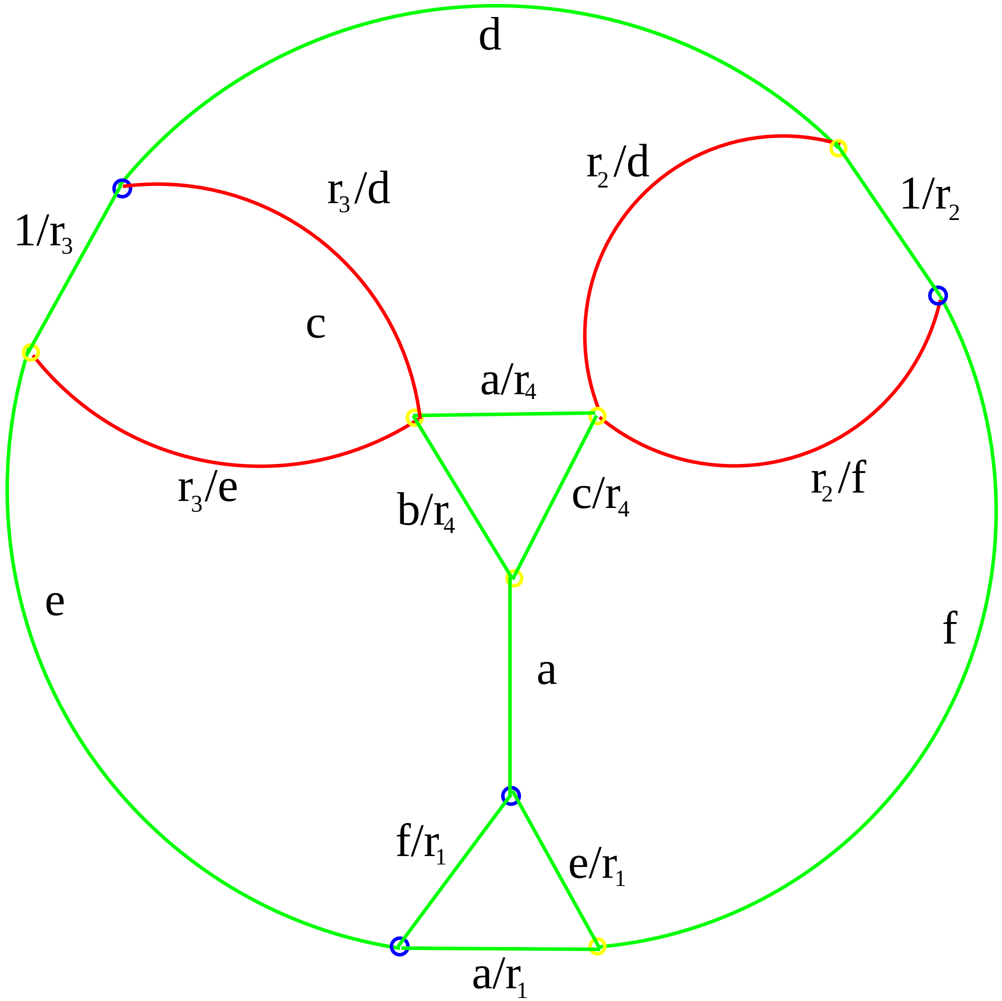
Another Cut Down

Notice there are 2^6 ways to choose our blue vertices...



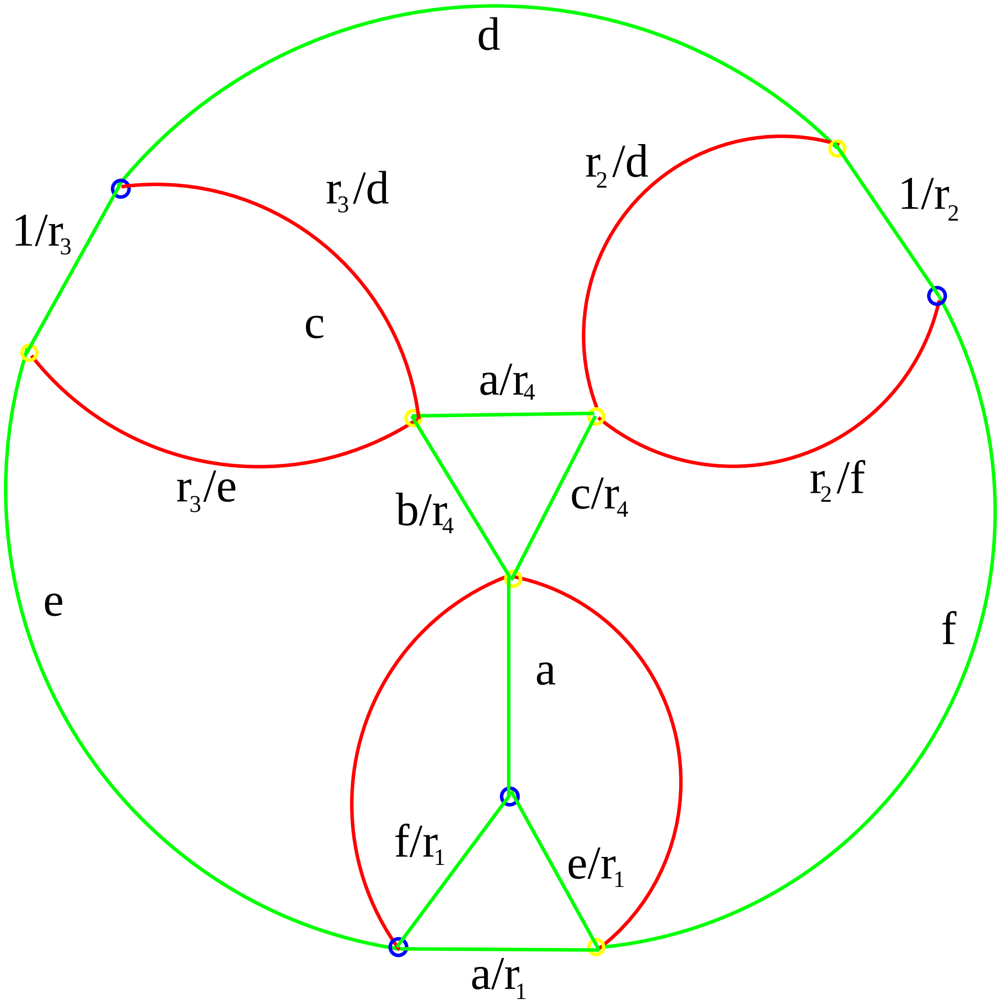
Another Cut Down

hence 2^6 way to perform a cut down.



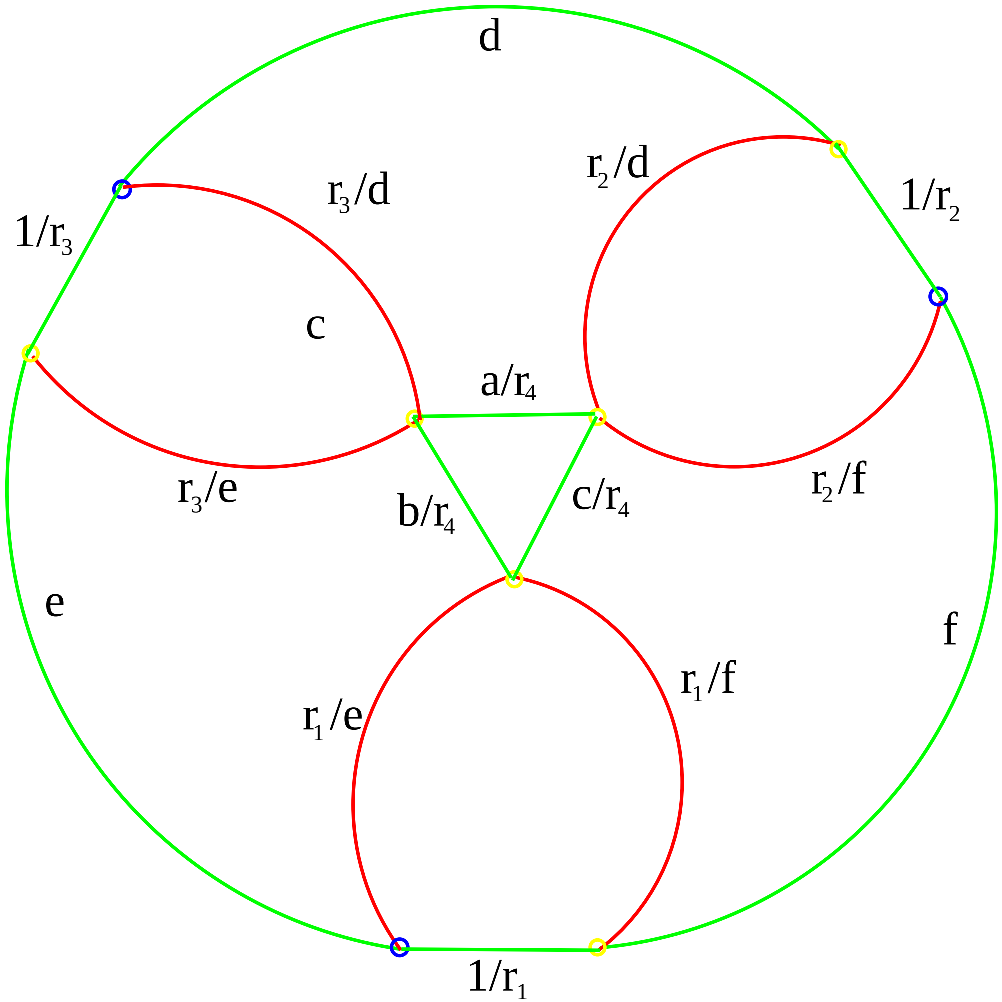
Another Cut Down

Though up to tetrahedral symmetries there are really only 4.



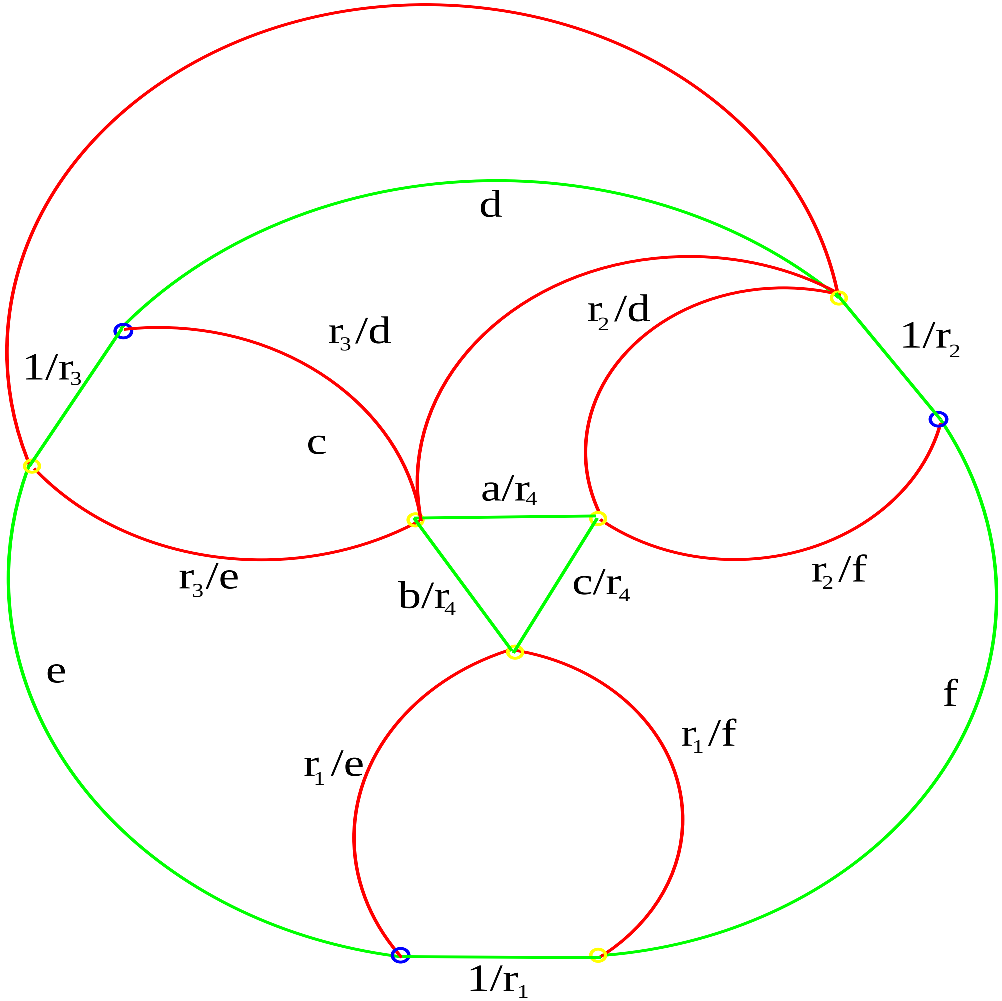
Another Cut Down

Last time we cut down we found exactly two green edges remained.



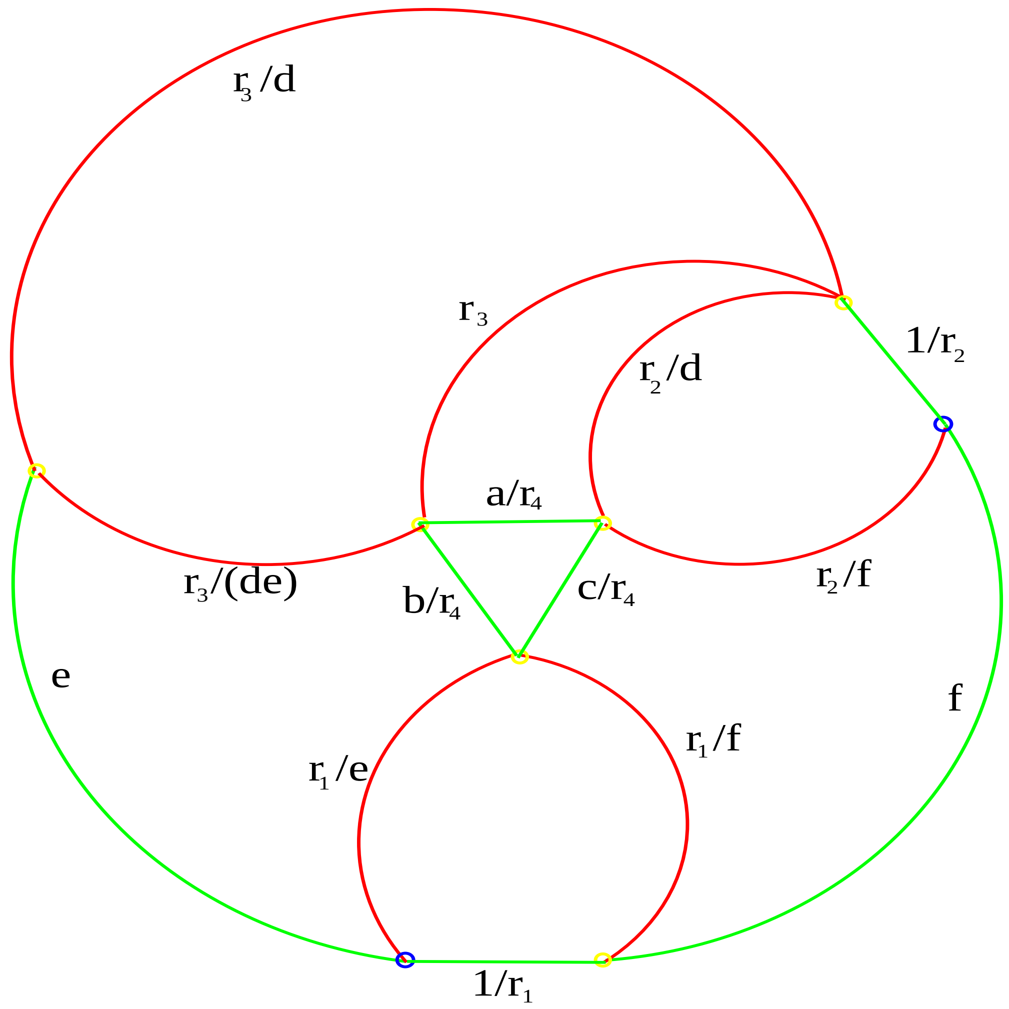
Another Cut Down

This time...



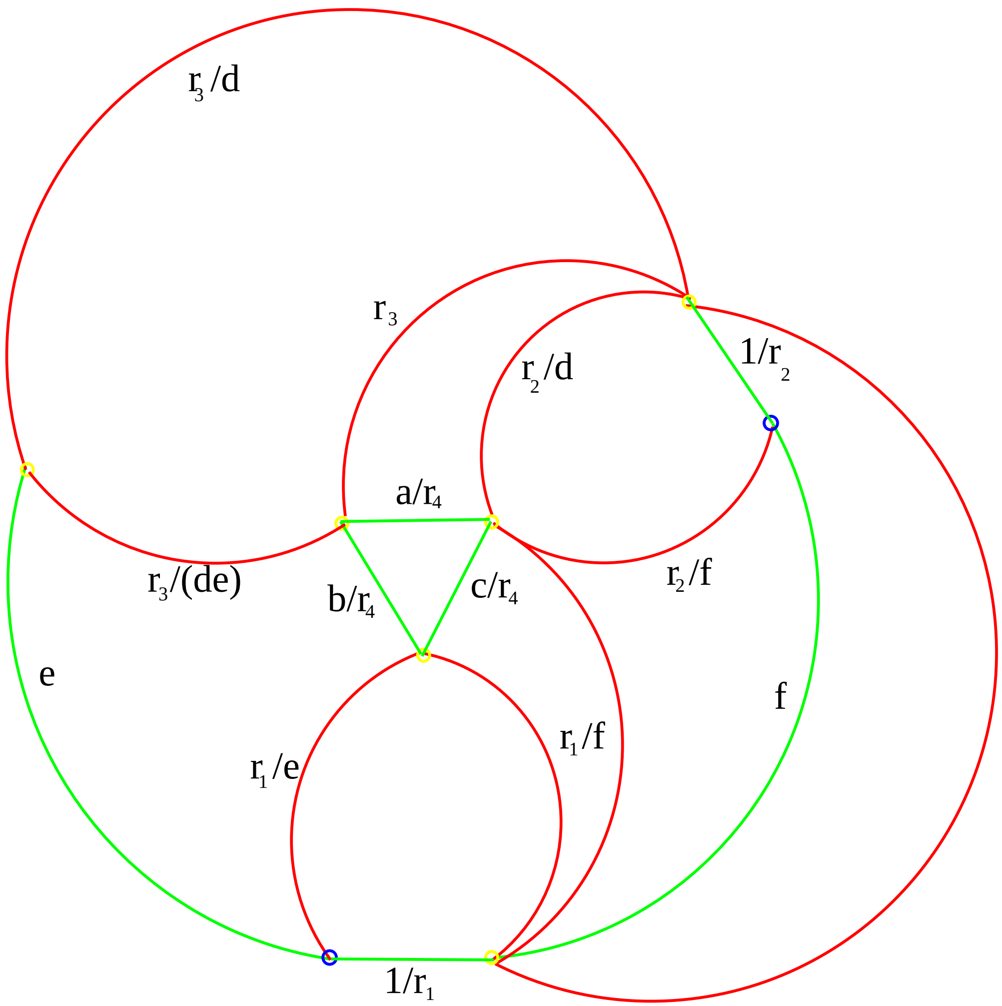
Another Cut Down

This time...



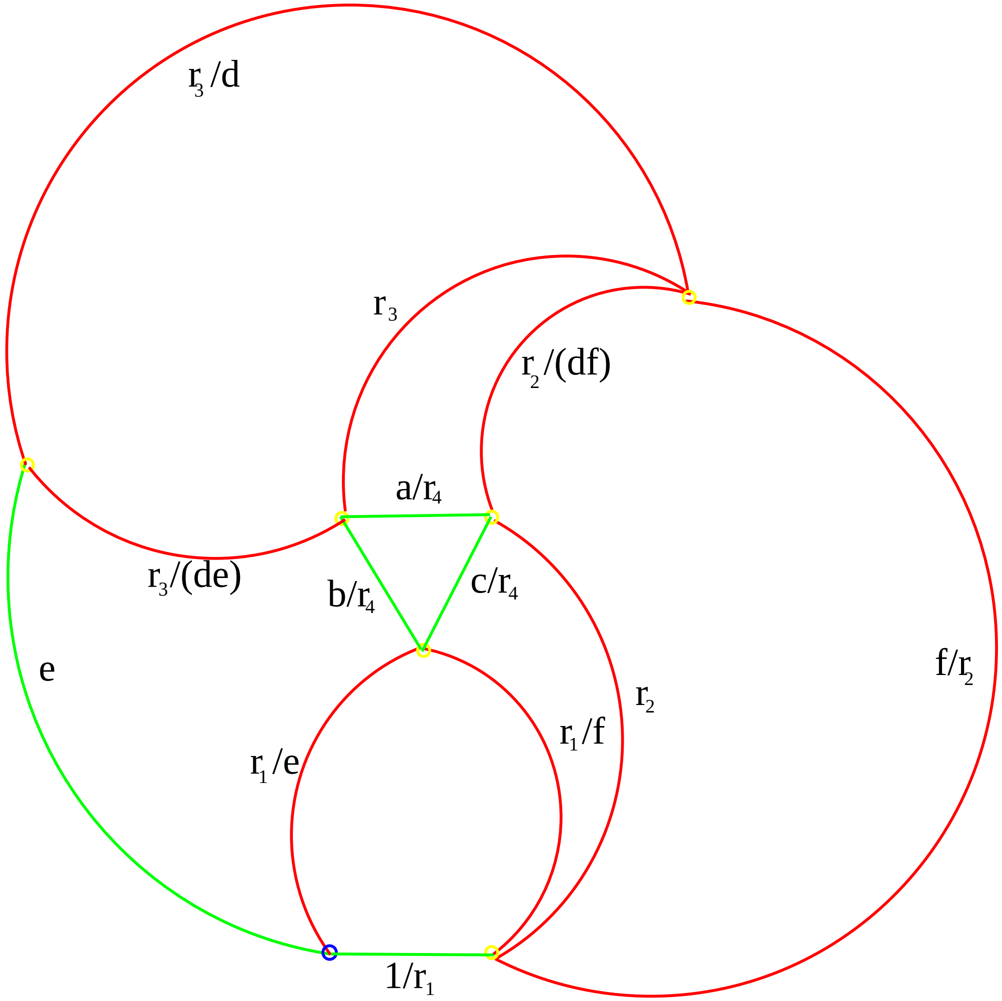
Another Cut Down

This time...



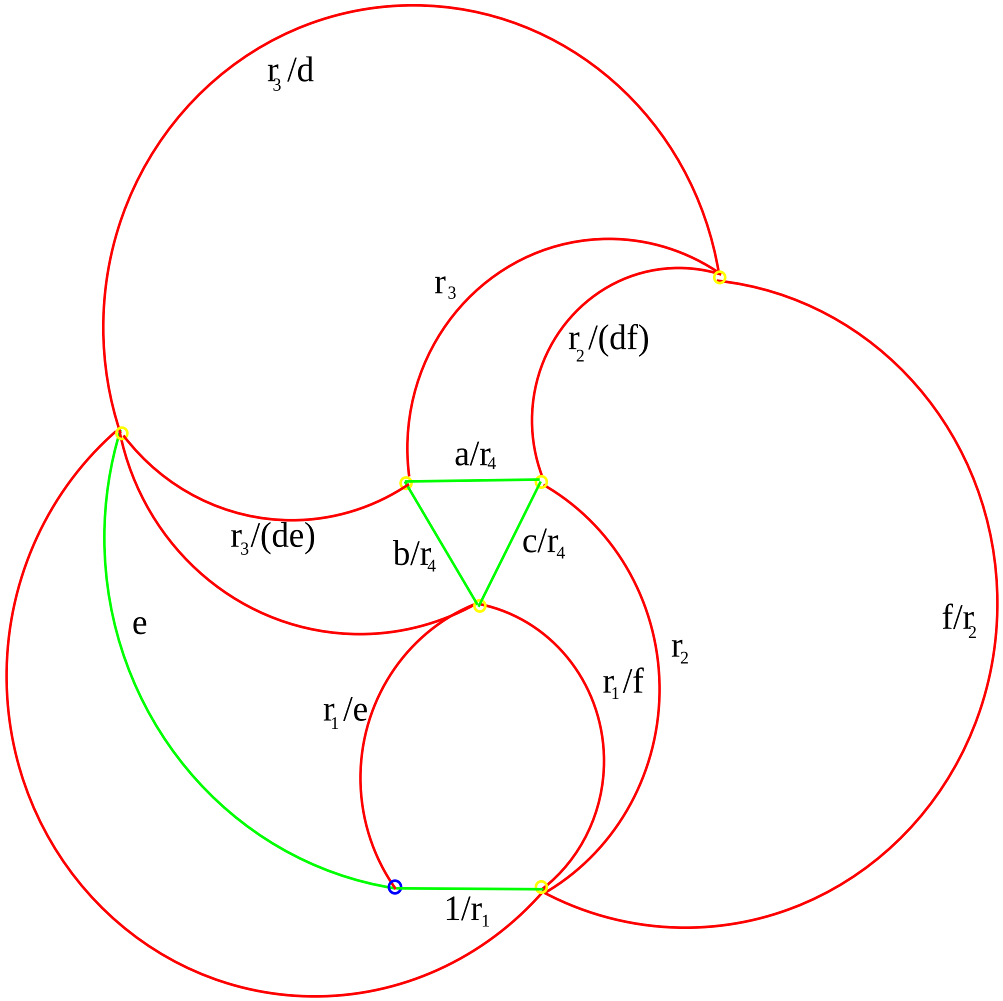
Another Cut Down

This time...



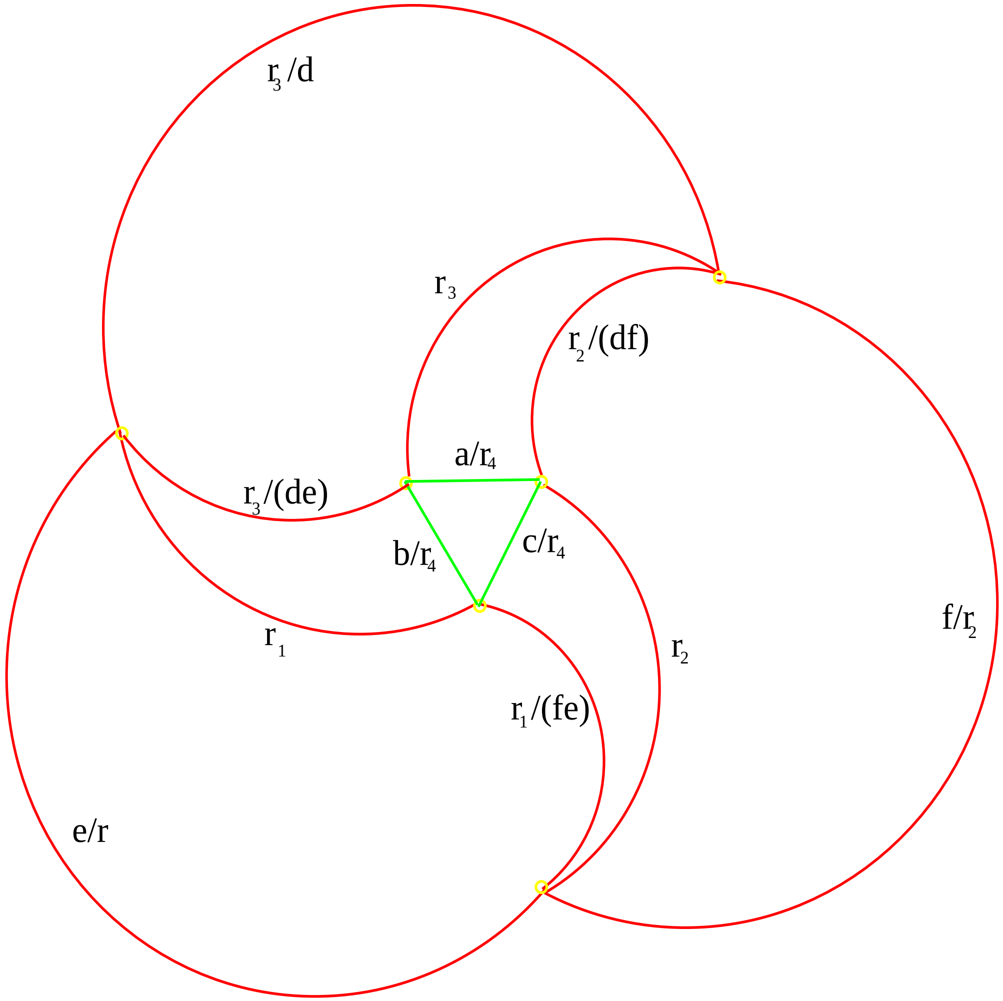
Another Cut Down

This time...



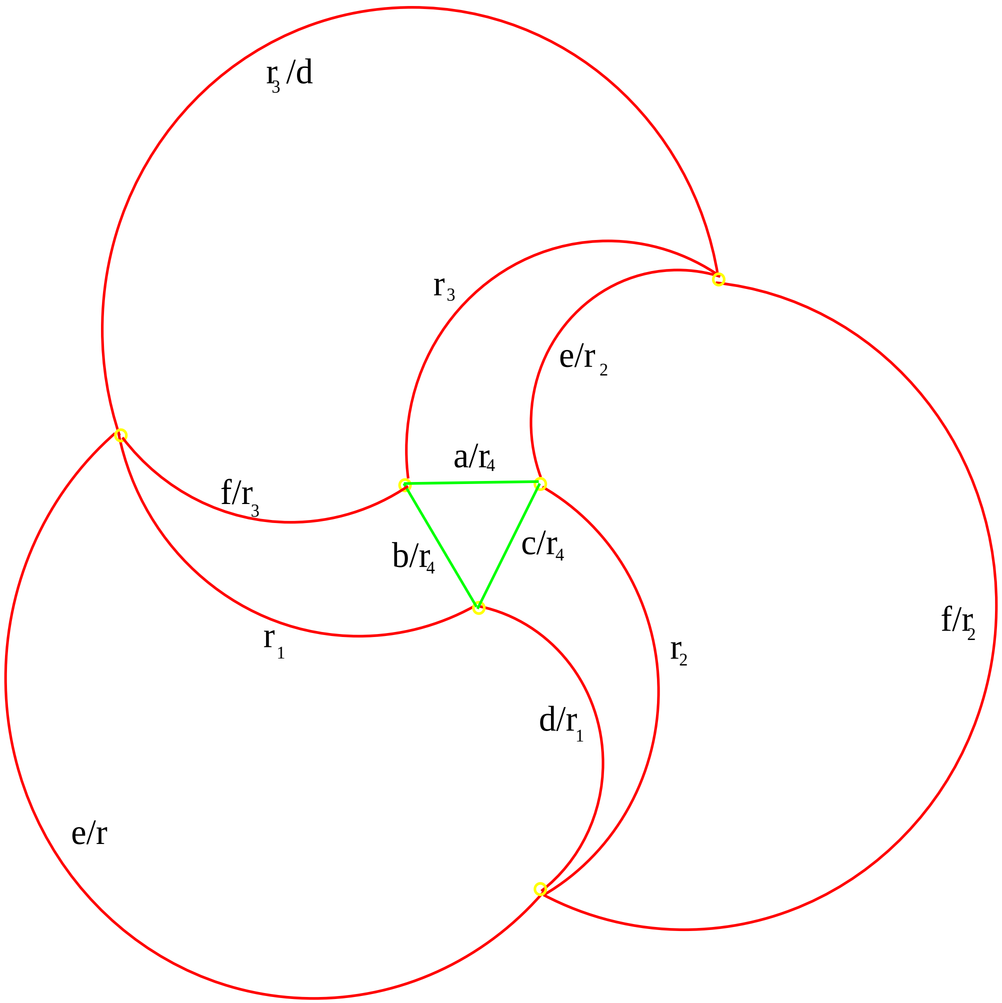
Another Cut Down

This time...



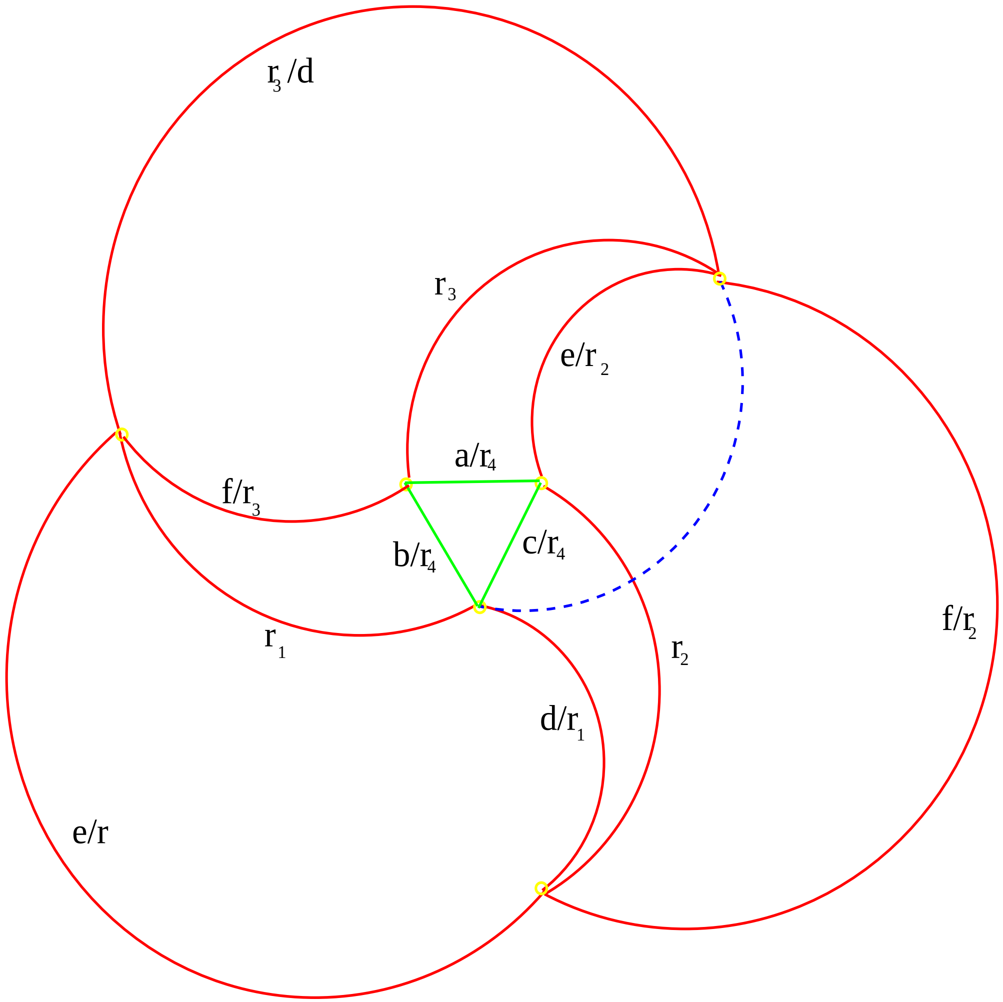
Another Cut Down

... we find there are three. Hence we have see experienced two of the four possible cut downs.



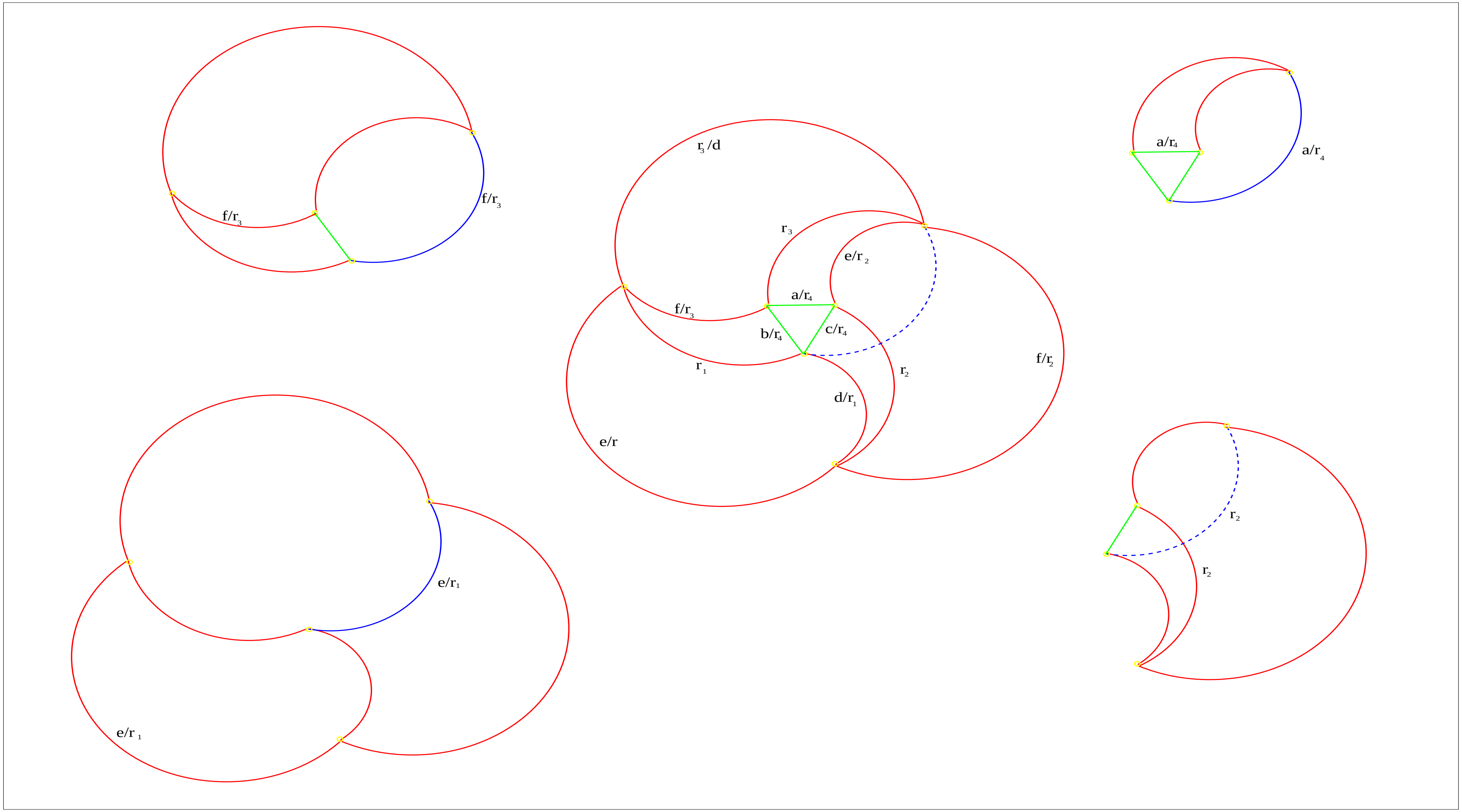
Another Cut Down

Once again let us pick a blue firepole.



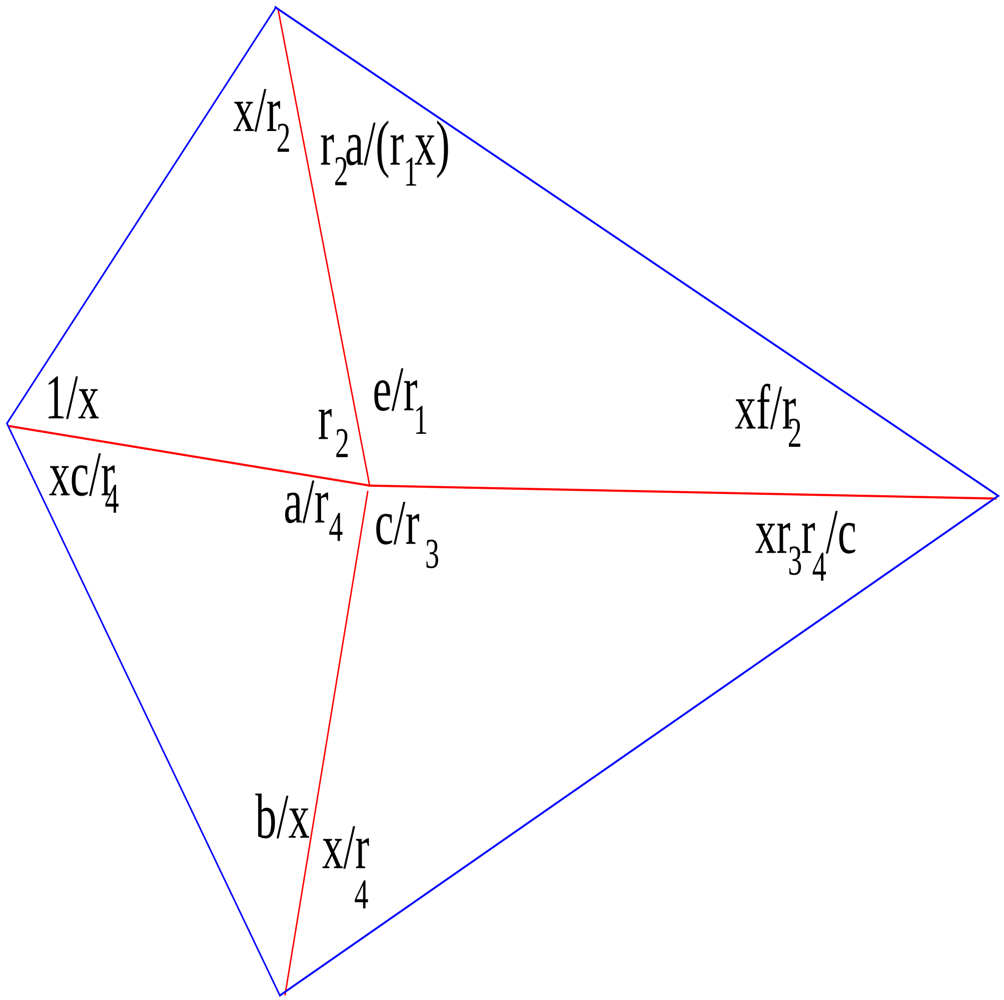
Another Cut Down

Notice we have four ideal tetrahedra.



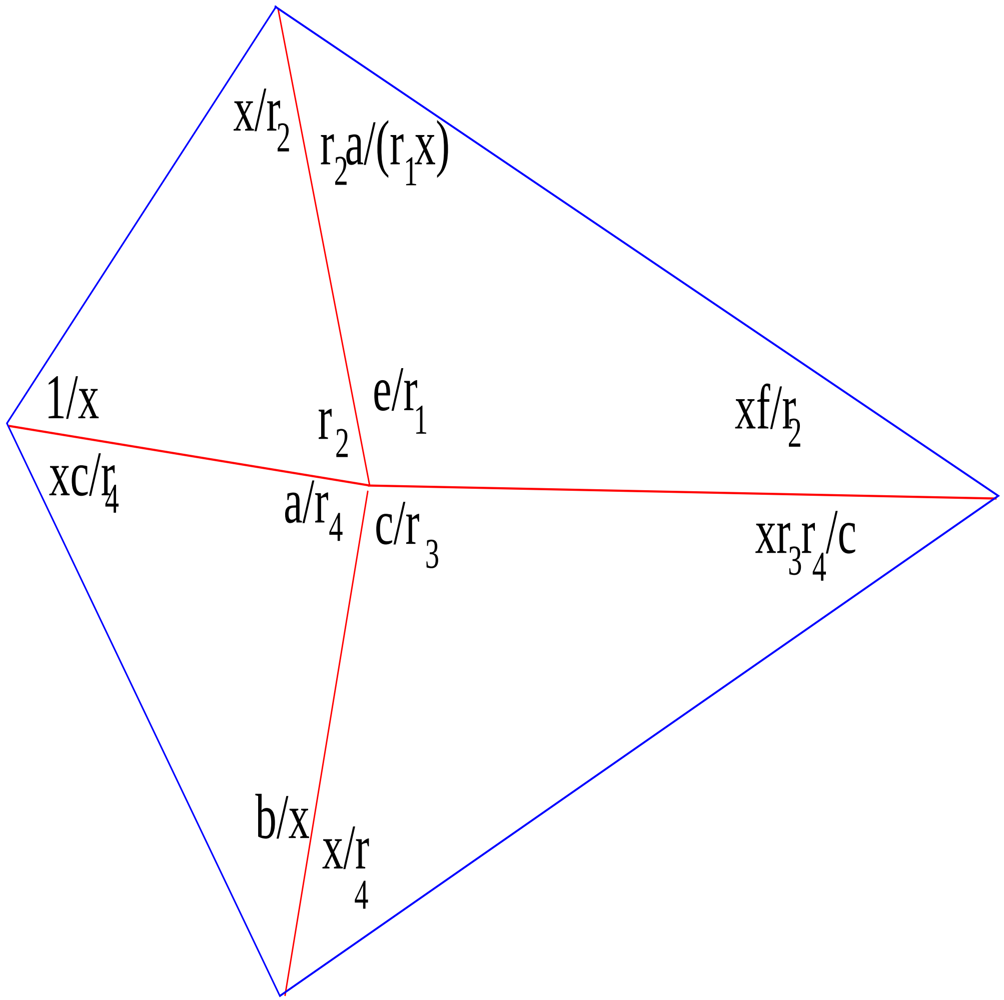
The Octahedron

To understand these four tetrahedra, we look down from the top of the fire pole and see:



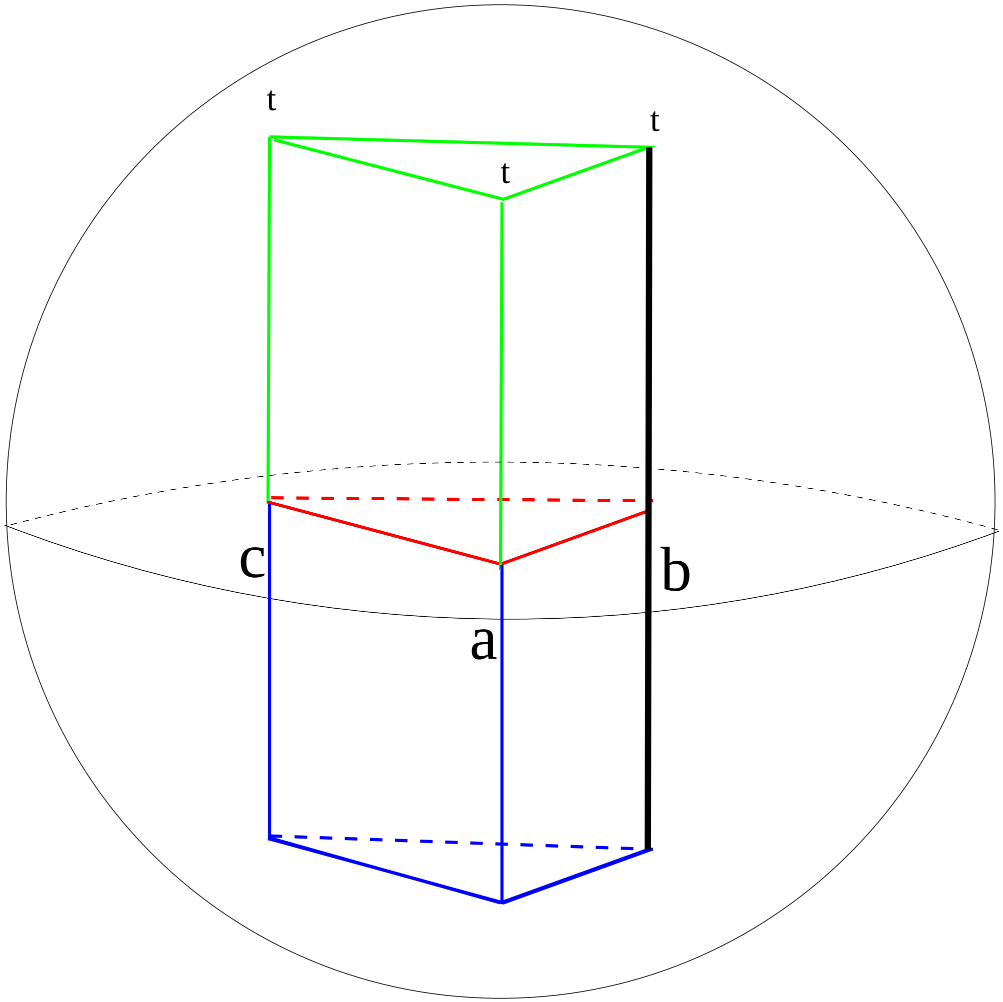
The Octahedron

Notice that the clinants are determined up to one unknown clinant x. Furthermore the z coordinates of our tetrahedra are determined by the clinants and satisfy $z_1 z_2 z_3 z_4 = 1$. Multiplying this out we find that x is determined by a quadratic polynomial.



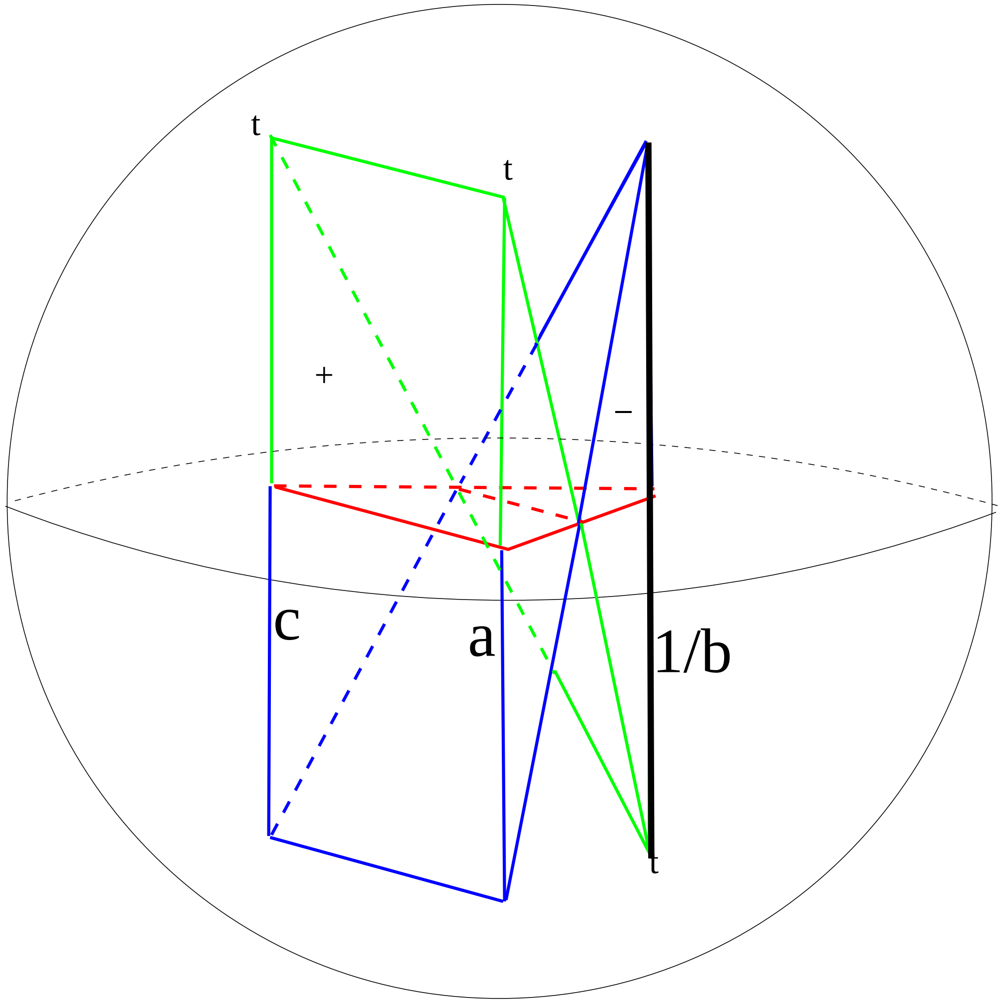
Flipping an Edge

This however, this quadratic is not the correct quadratic. To find the correct quadratic, we must understand the result of flipping the black edge.



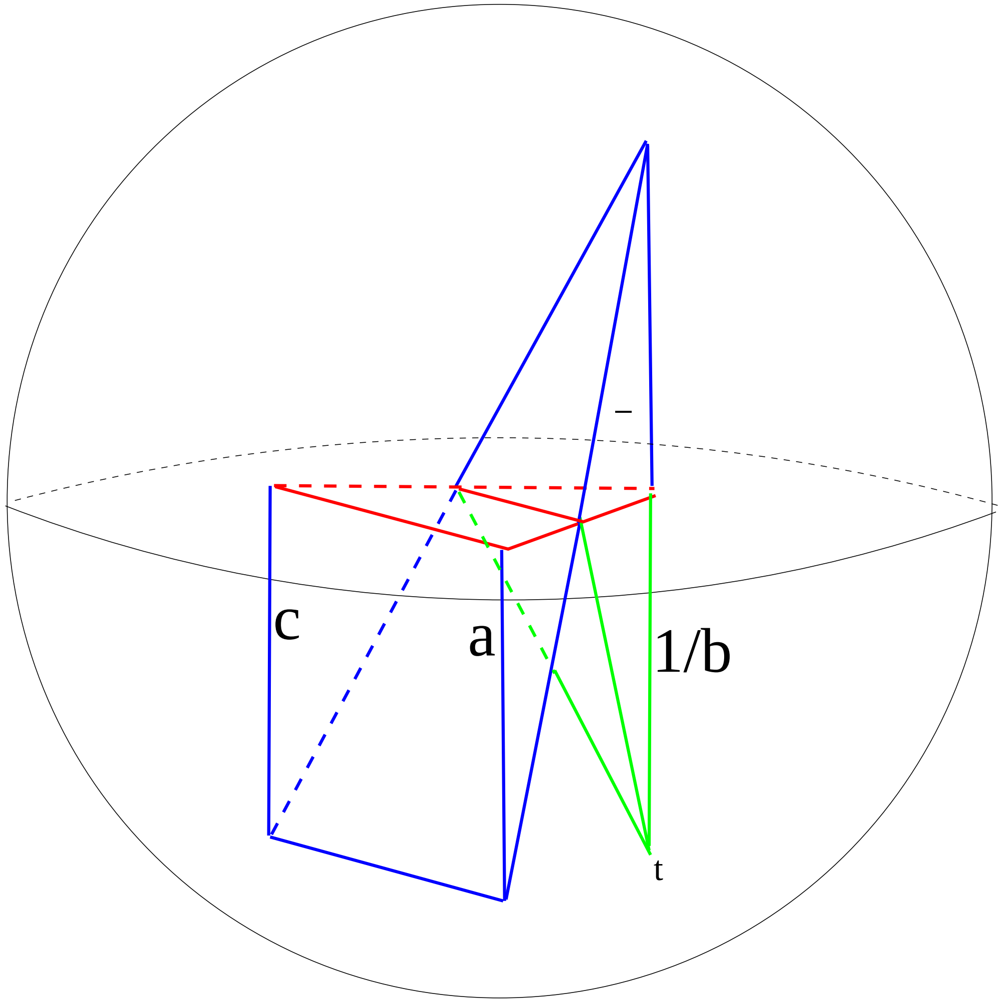
Flipping an Edge

We find:



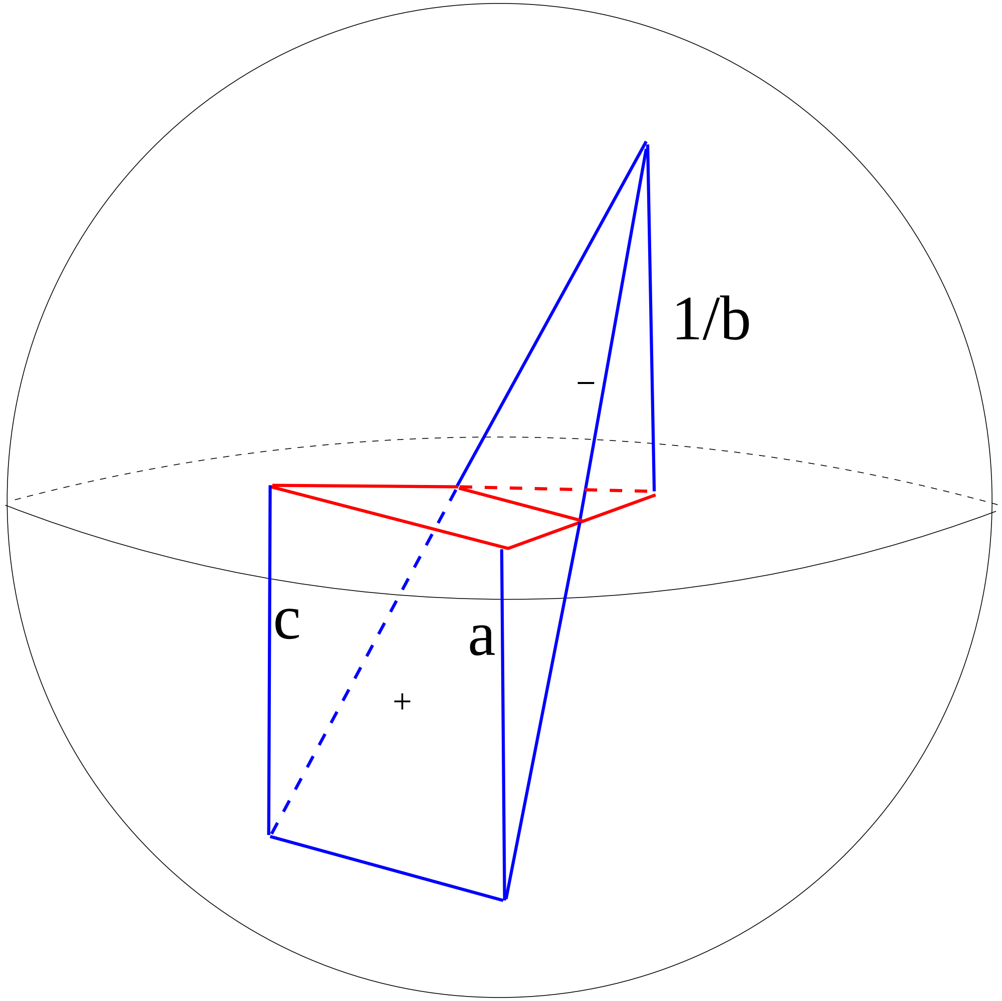
Flipping an Edge

We can still remove the top half. One piece..



Flipping an Edge

...then the other.

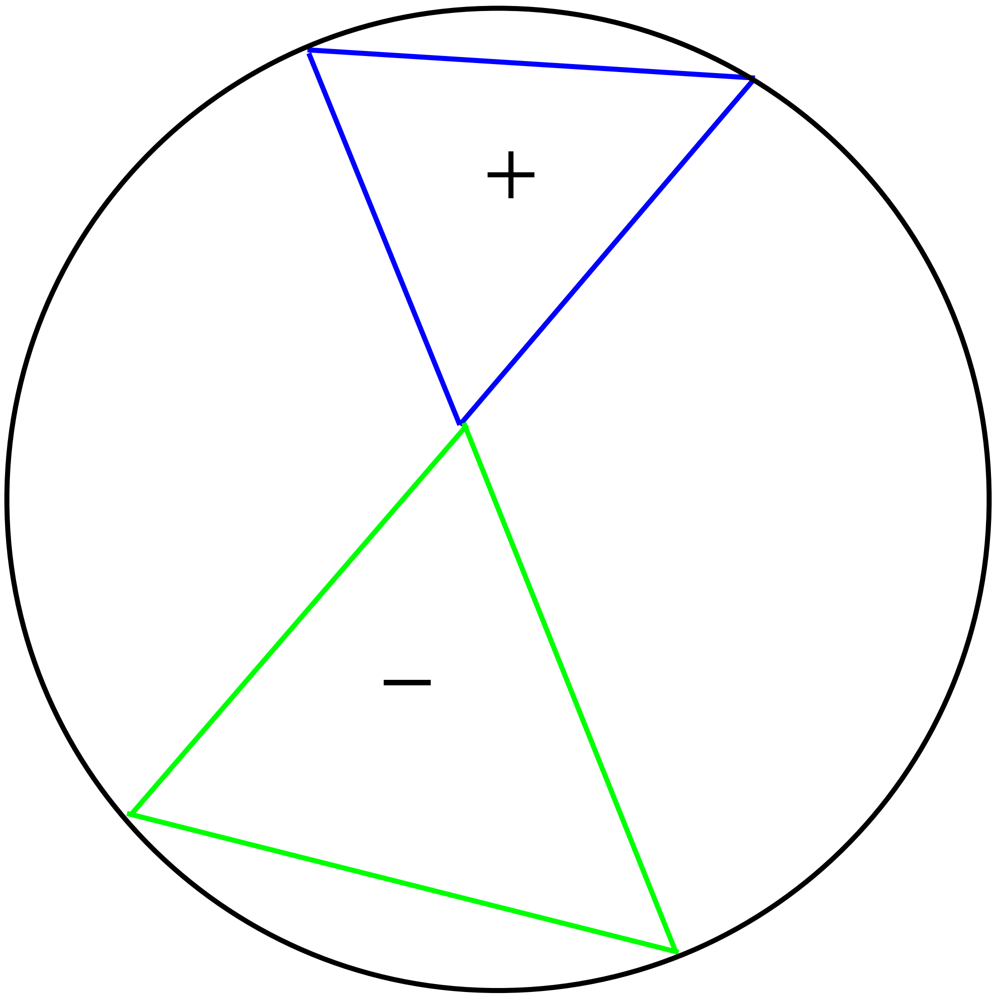


Tetrahedron Preservation

Theorem: Flipping a supertetrahedral edge preserves the underlying tetrahedron

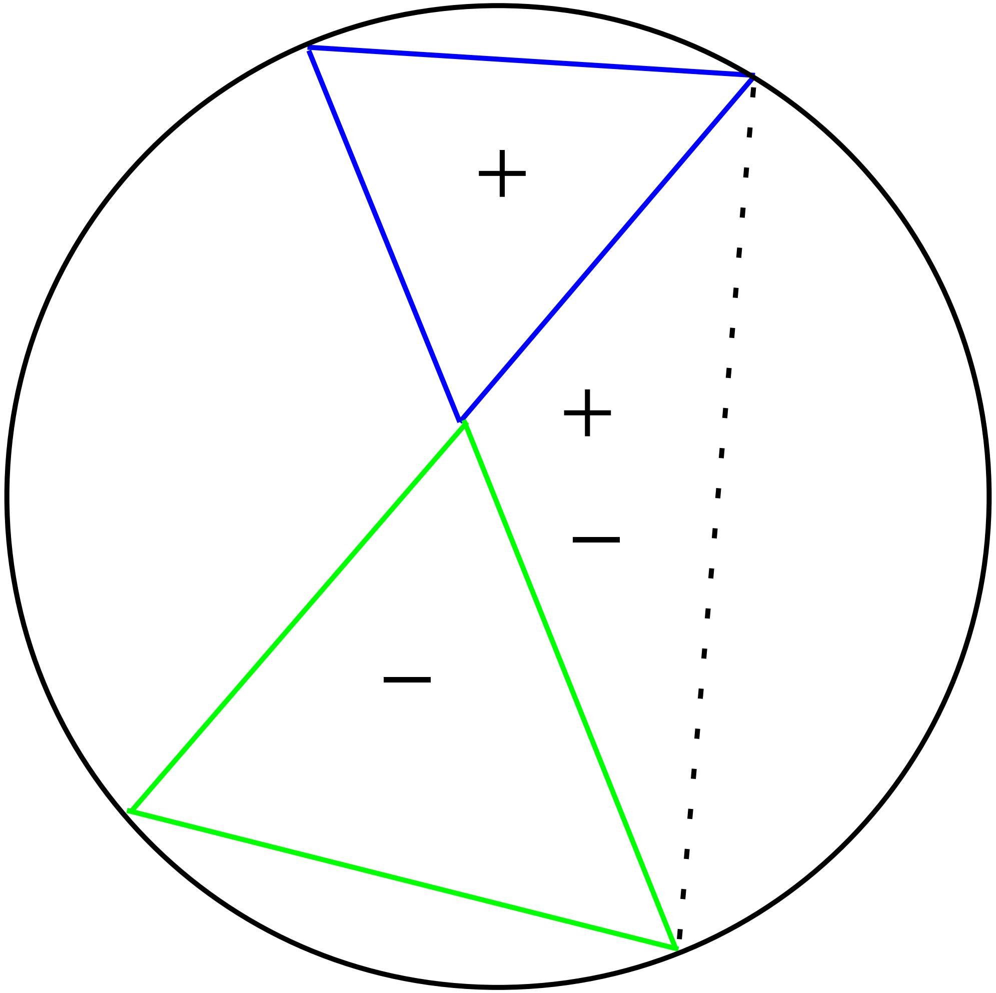
Flipping Preserves the Triangle

To see this we will demonstrate this theorem in two dimensions. We start by looking at the prism.



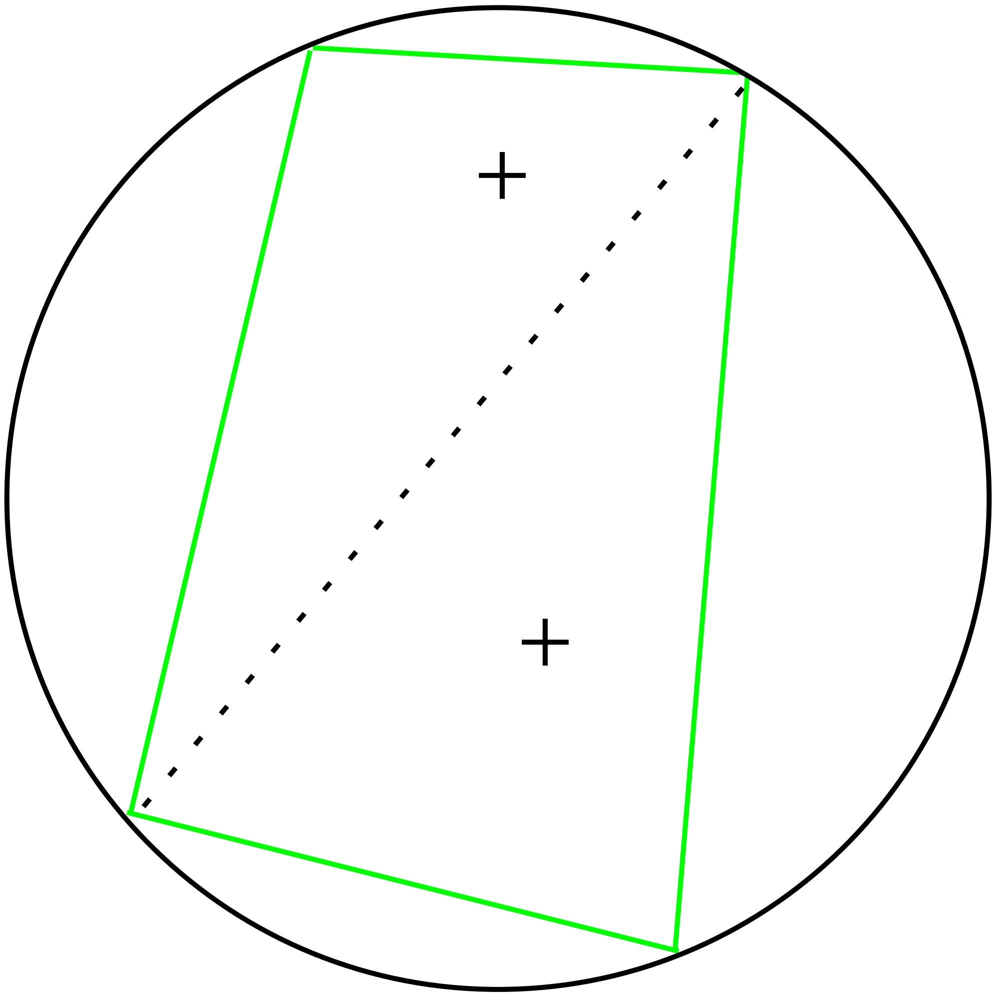
Flipping Preserves the Triangle

Notice the prism is built of ideal triangles. As a scissors class it is 0.



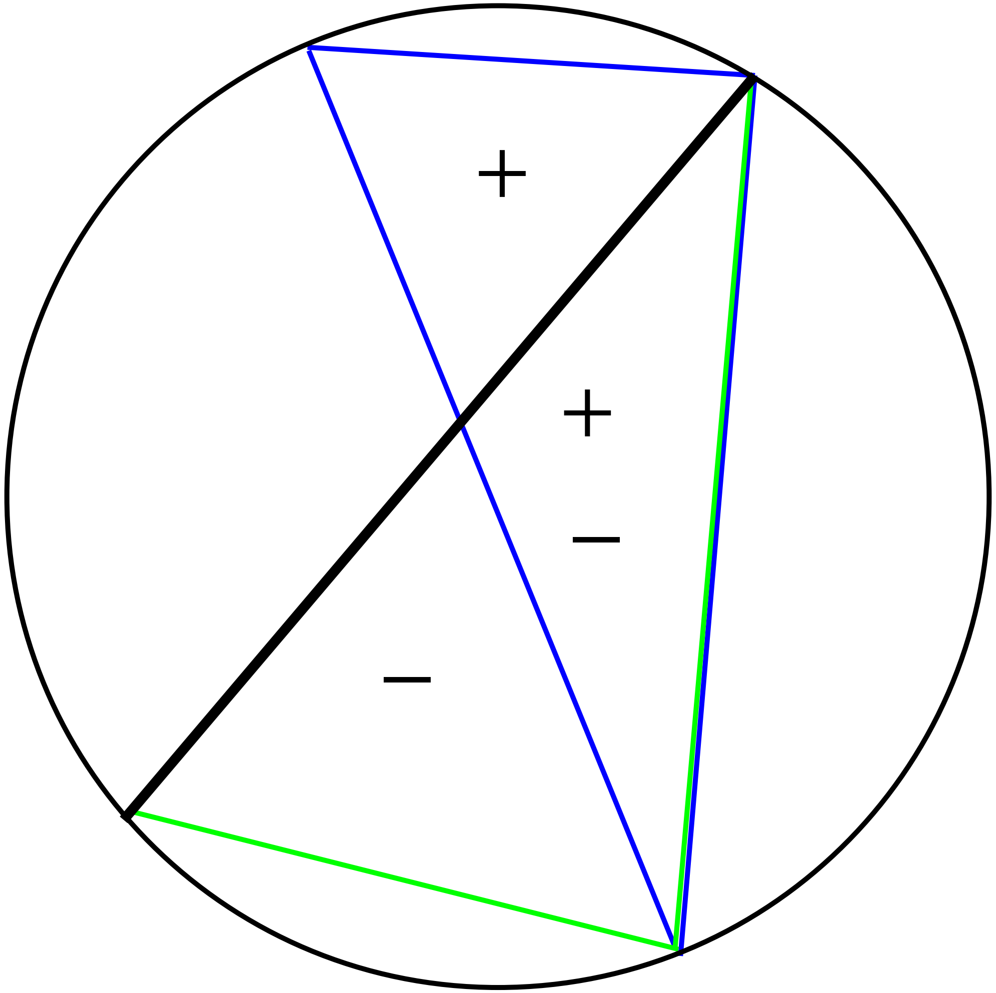
Flipping Preserves the Triangle

Here is the untwisted 2 dimensional prism. Notice its scissors class is $2[I] \neq 0$.



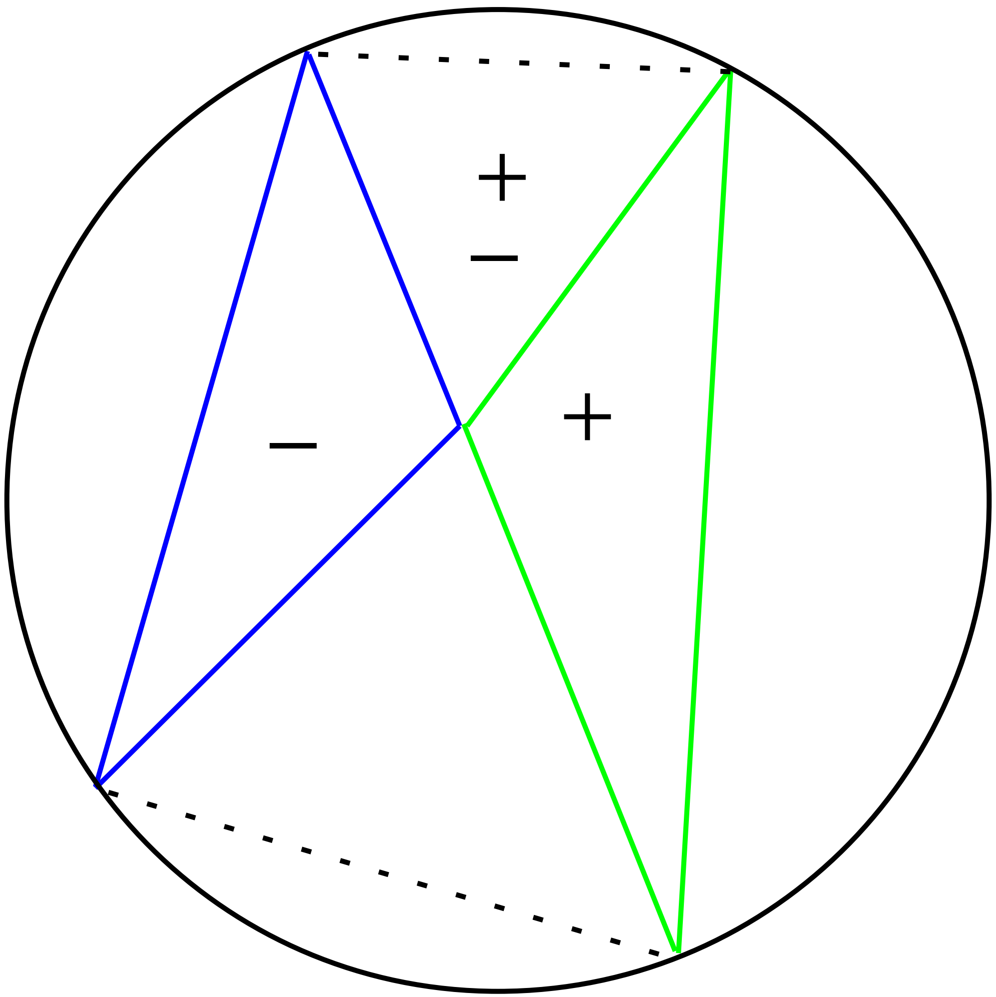
Flipping Preserves the Triangle

Maybe this view will help. We will flip the black edge.



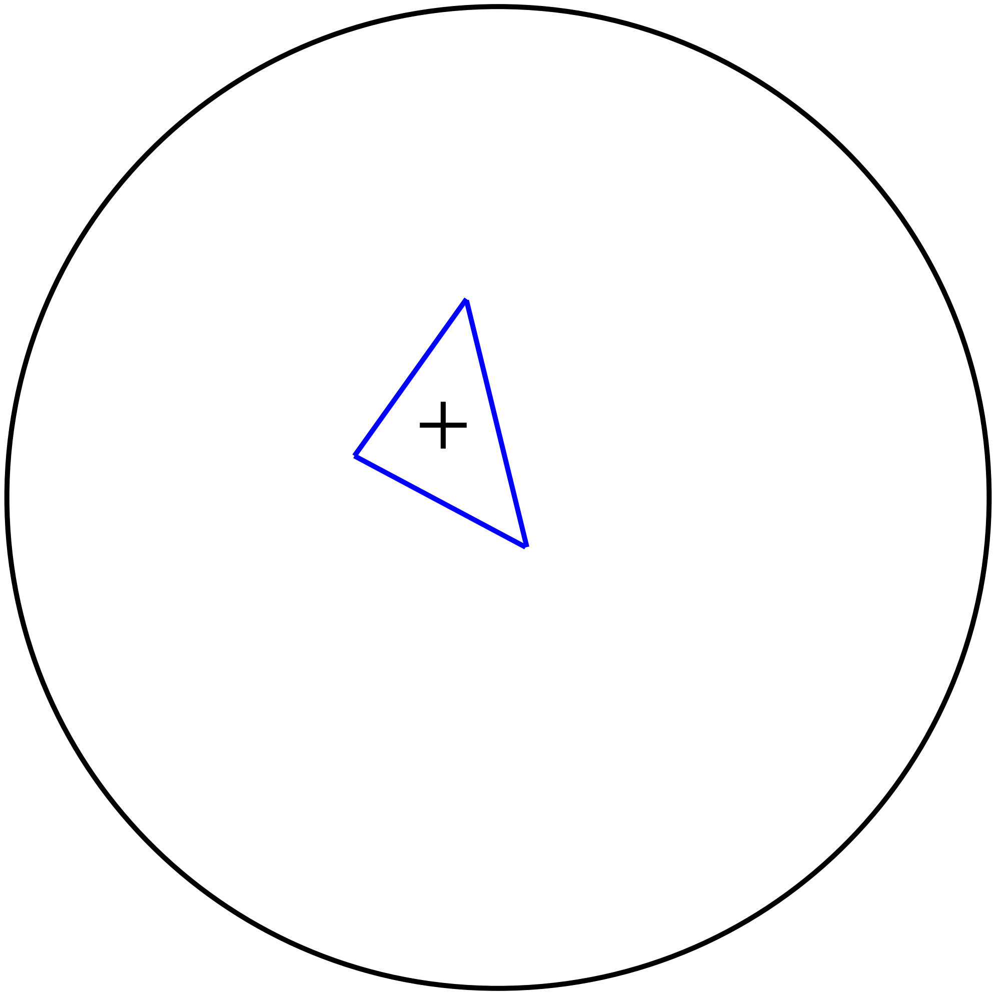
Flipping Preserves the Triangle

We find this region.



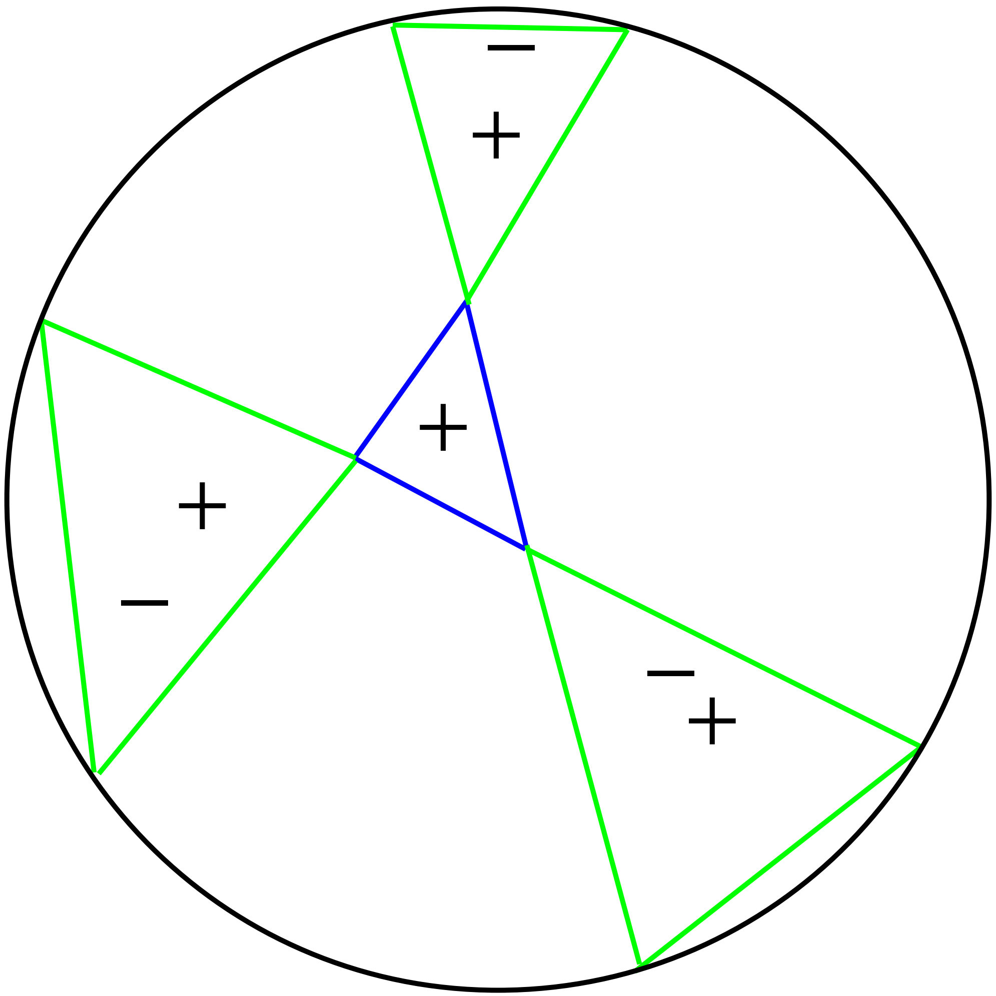
Flipping Preserves the Triangle

Now take a triangle.



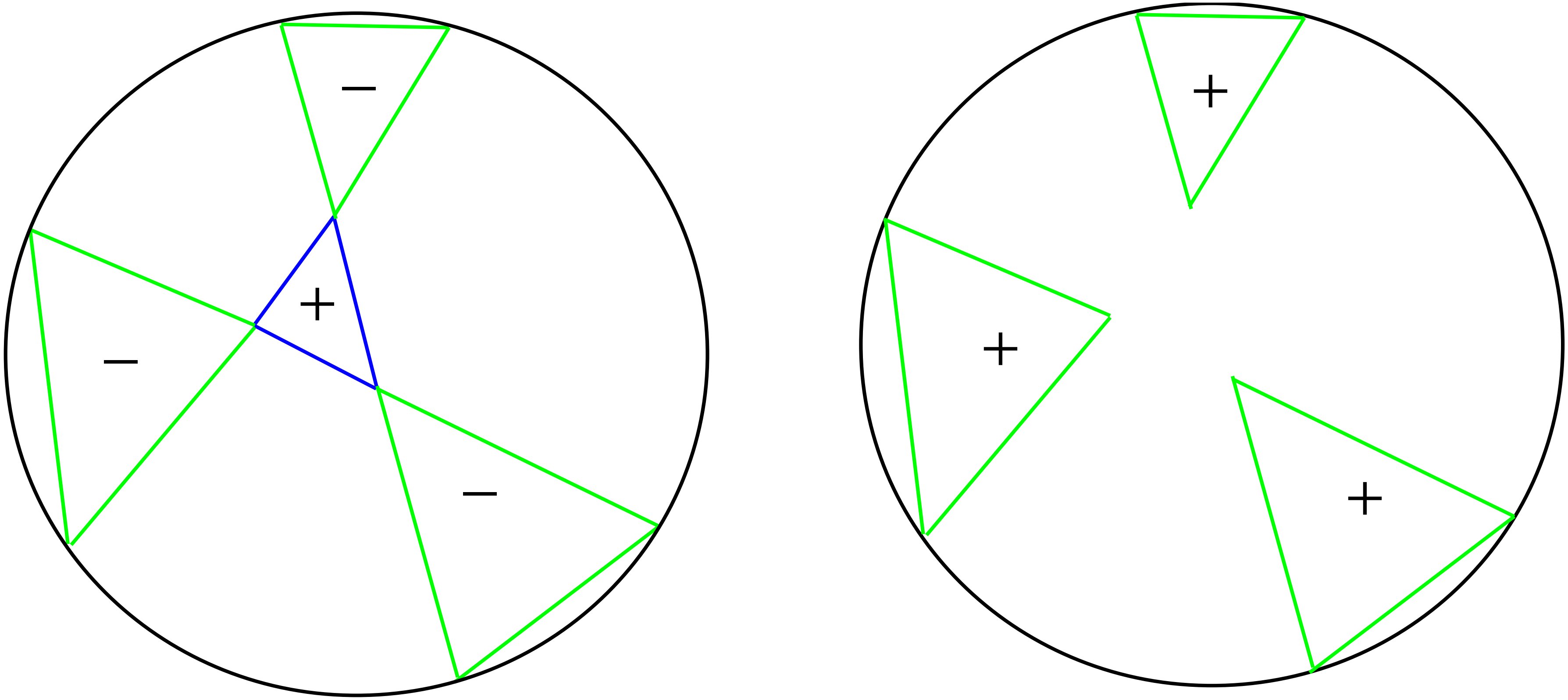
Flipping Preserves the Triangle

Let us view our triangle as a supertrianlge with three half prism removed.



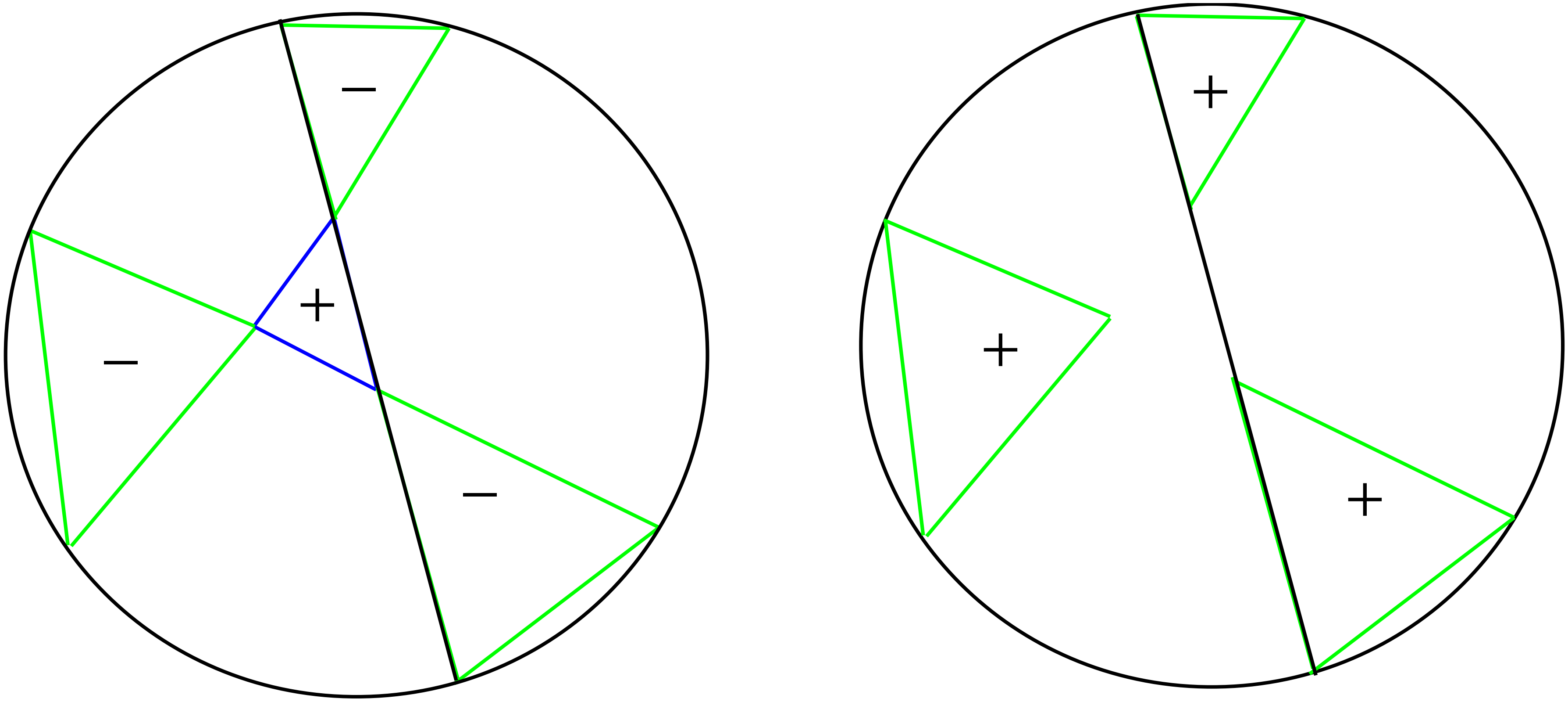
Flipping Preserves the Triangle

Let us view it as such:



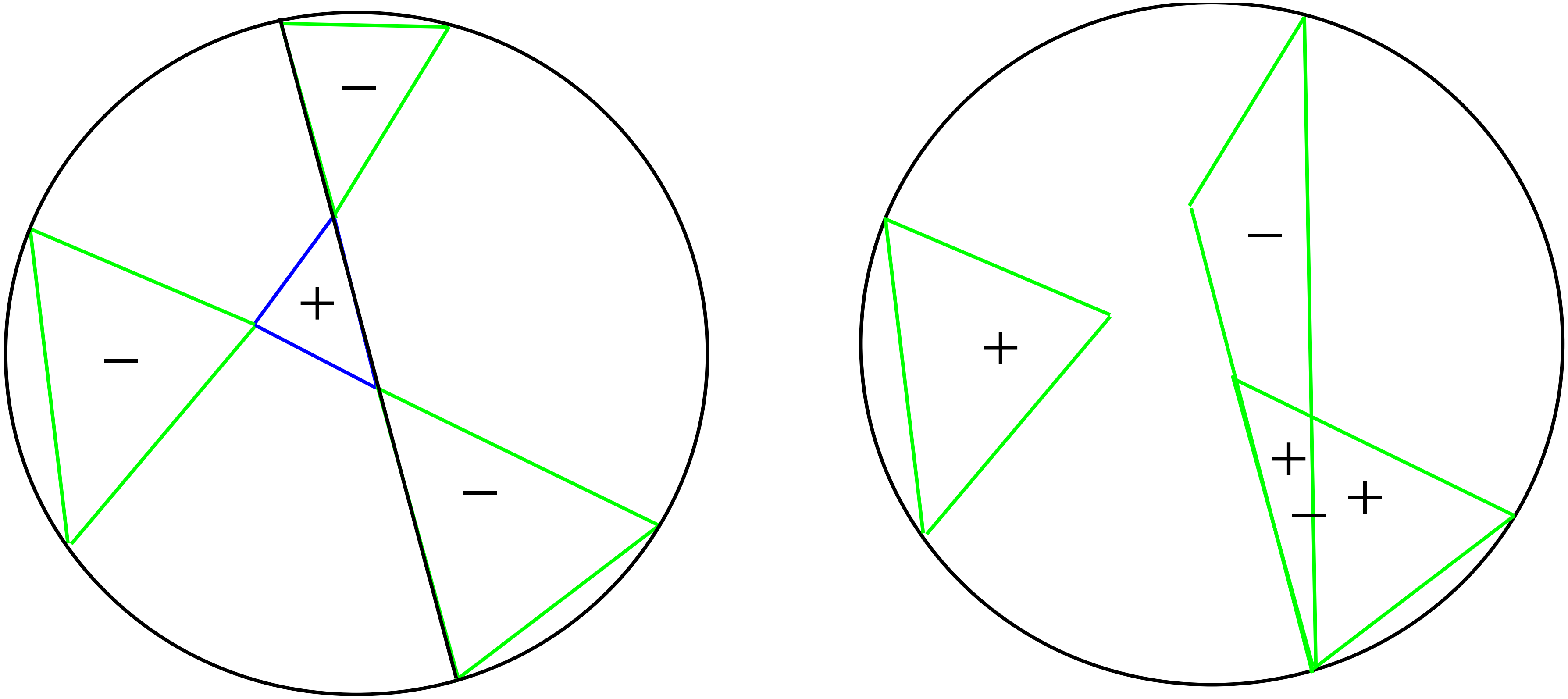
Flipping Preserves the Triangle

We will flip the black edge.



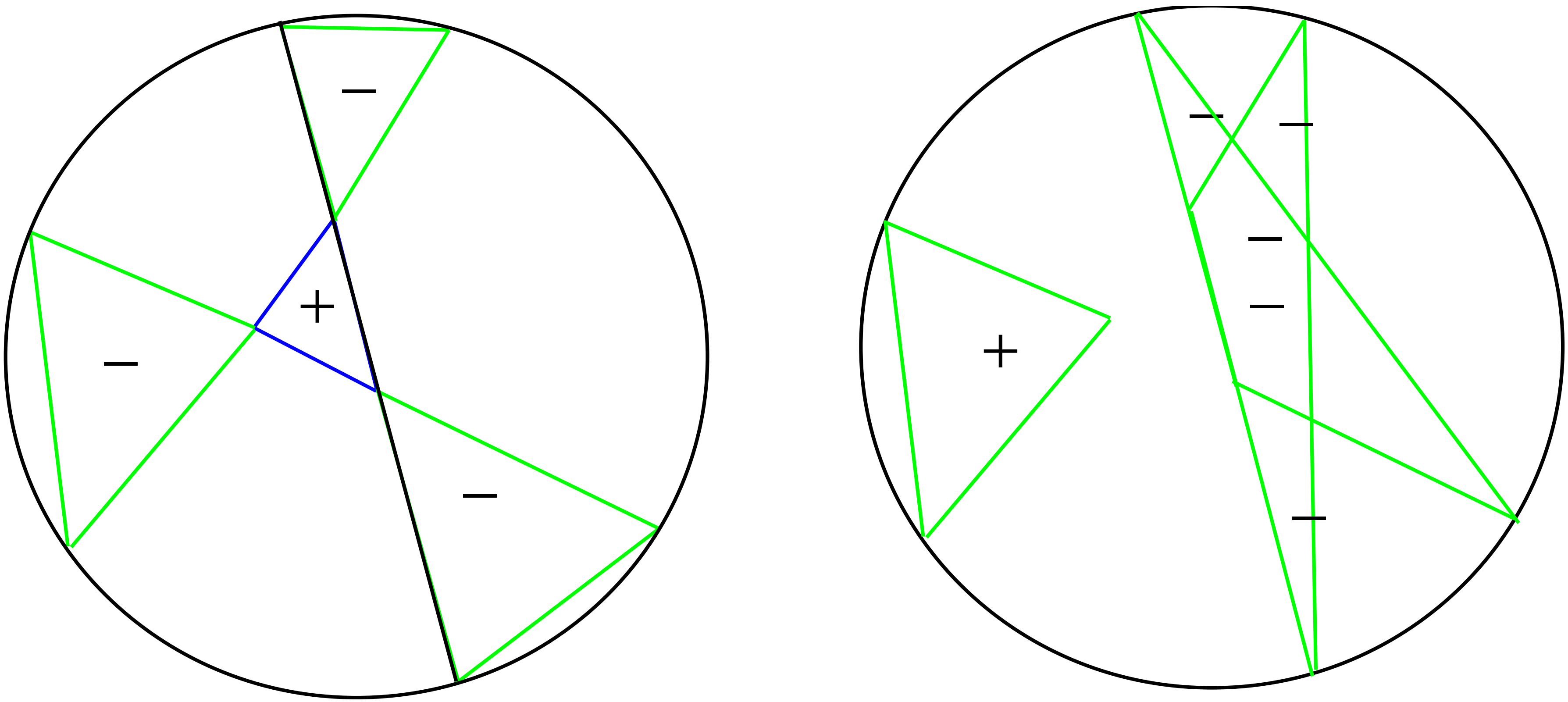
Flipping Preserves the Triangle

Let us flip one prism at a time. The first...



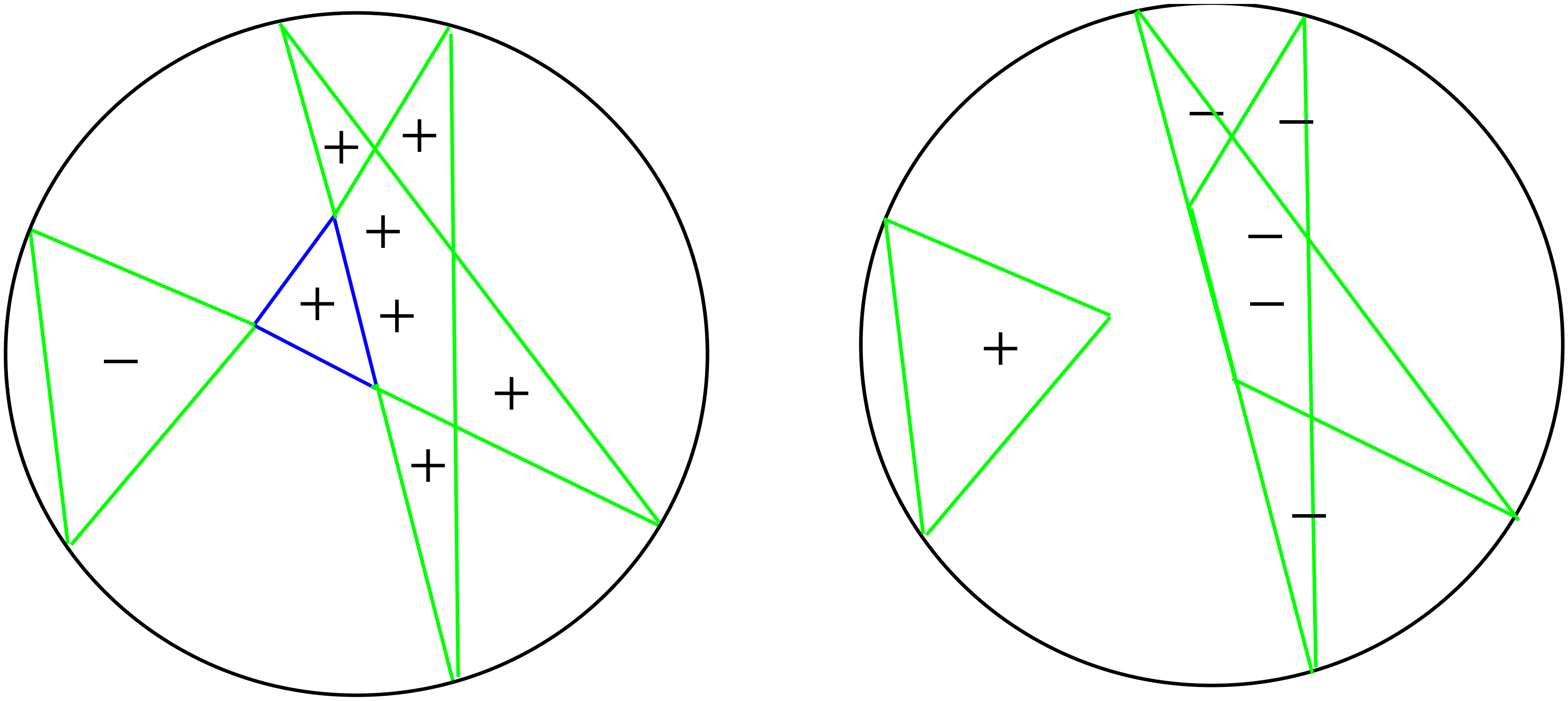
Flipping Preserves the Triangle

... and the second.



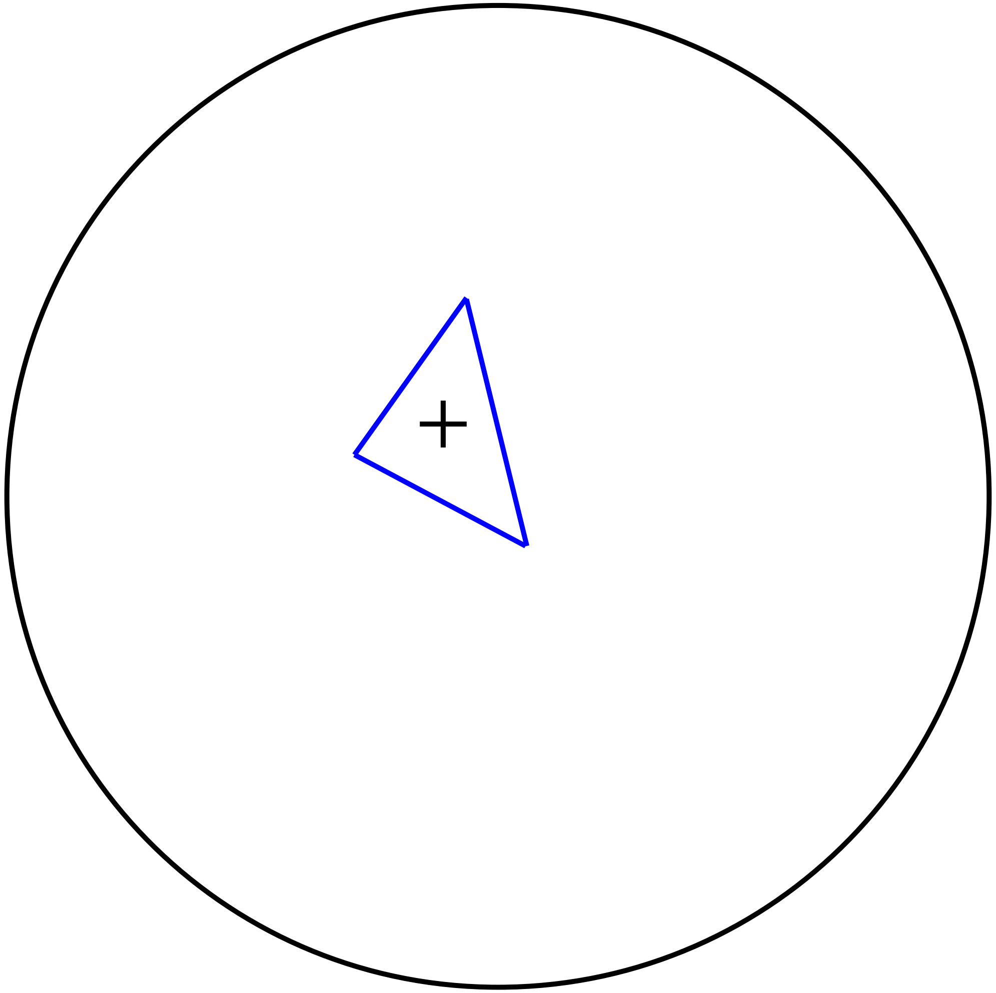
Flipping Preserves the Triangle

Now we flip the supertrianlge. Once again, we can simply keep track o fan ideal decomposition of the supertrianlge to do this.



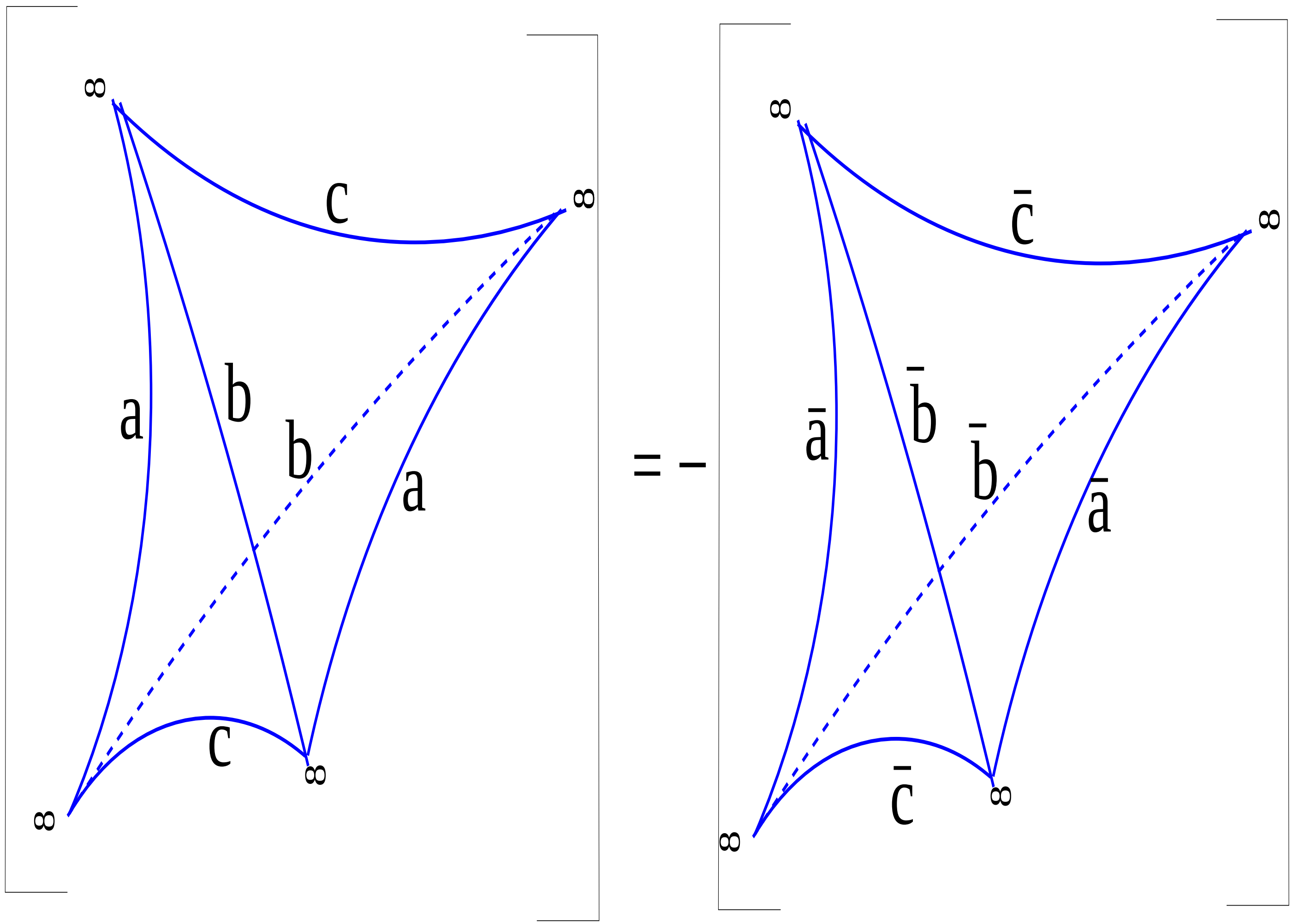
Flipping Preserves the Triangle

Combining these regions results in the need triangle.



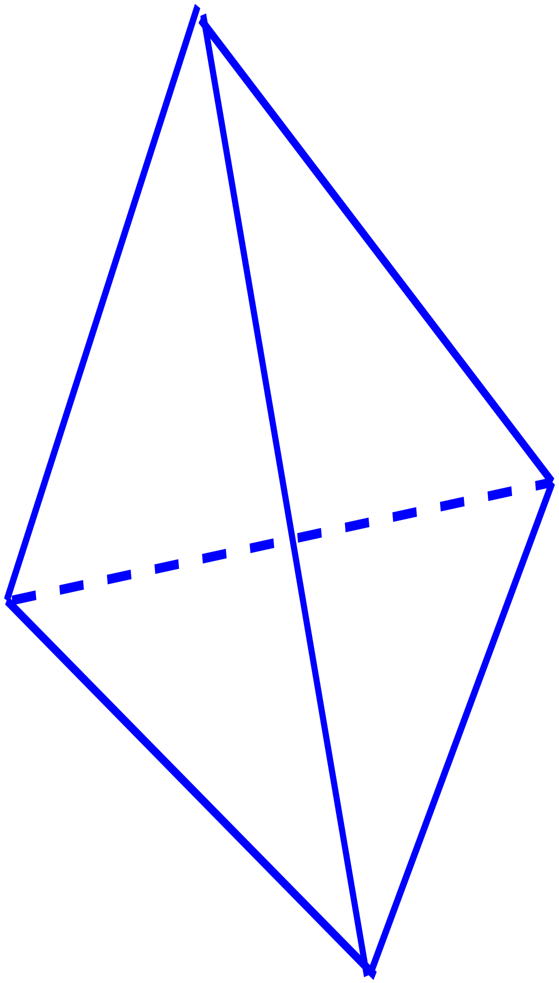
A Volume Formula

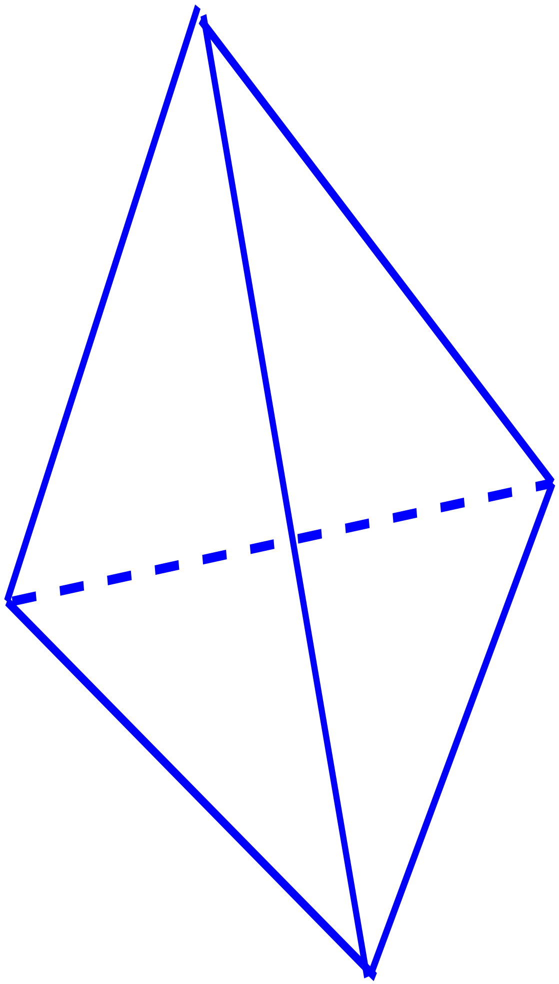
One thing to note about flipping an edge flip is that it conjugates the clinant along the flipped edge. When using the flip its is useful to keep in mind that that:



A Volume Formula

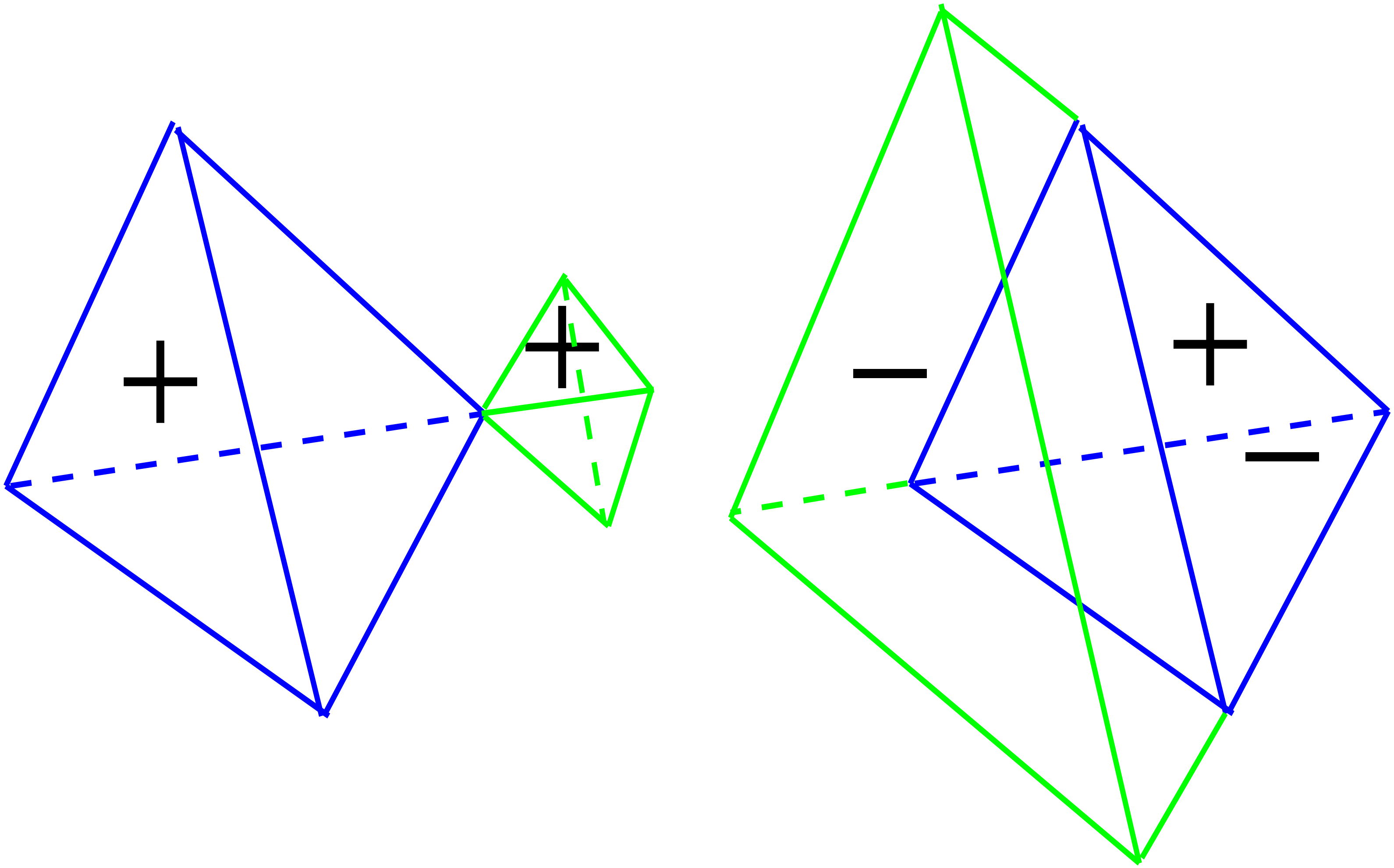
Flipping pays off when we look at a doubled tetrahedron.





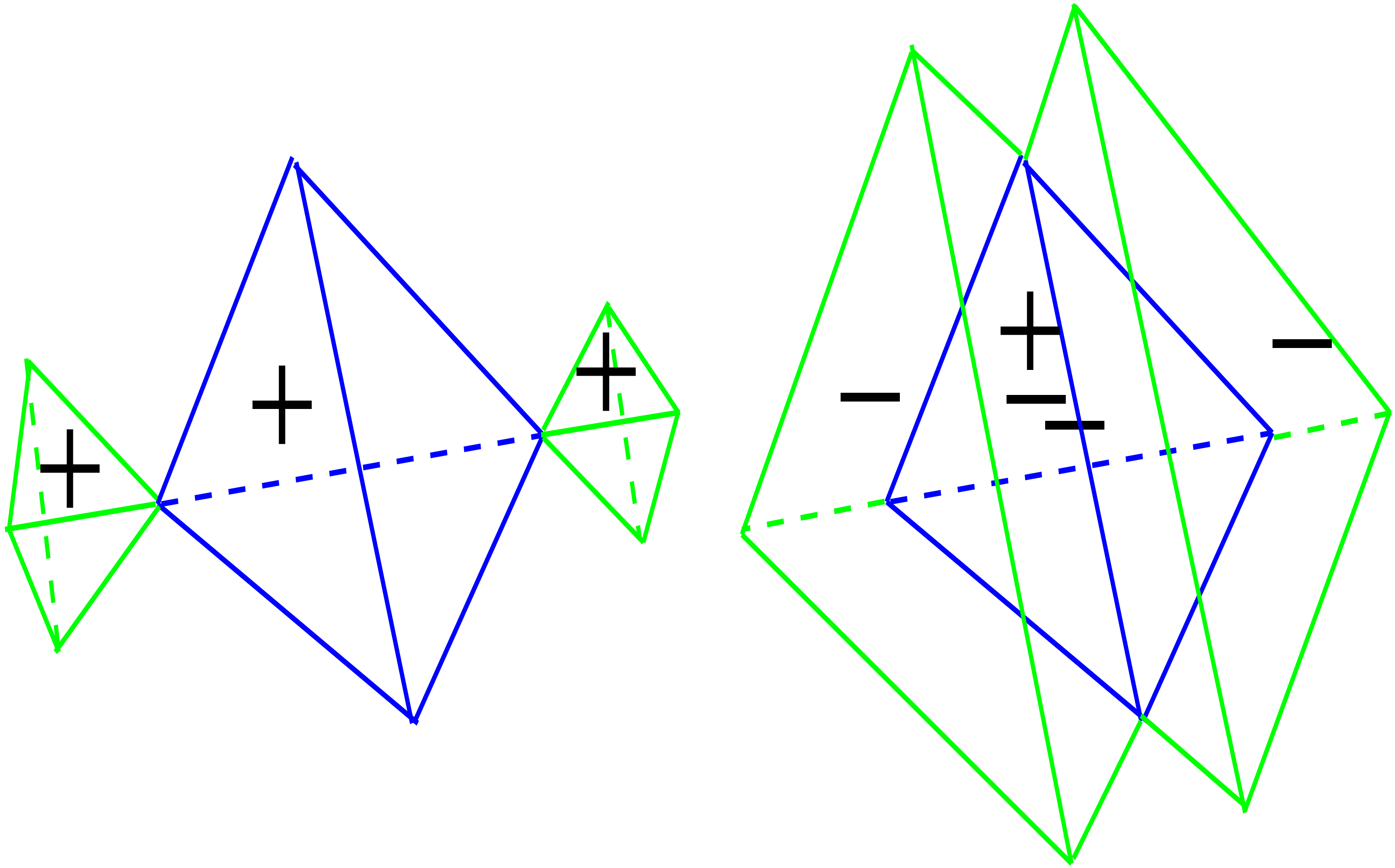
A Volume Formula

Adjoin a twist tie to one vertex and it's flipped image to other. As a scissors class you've done nothing!



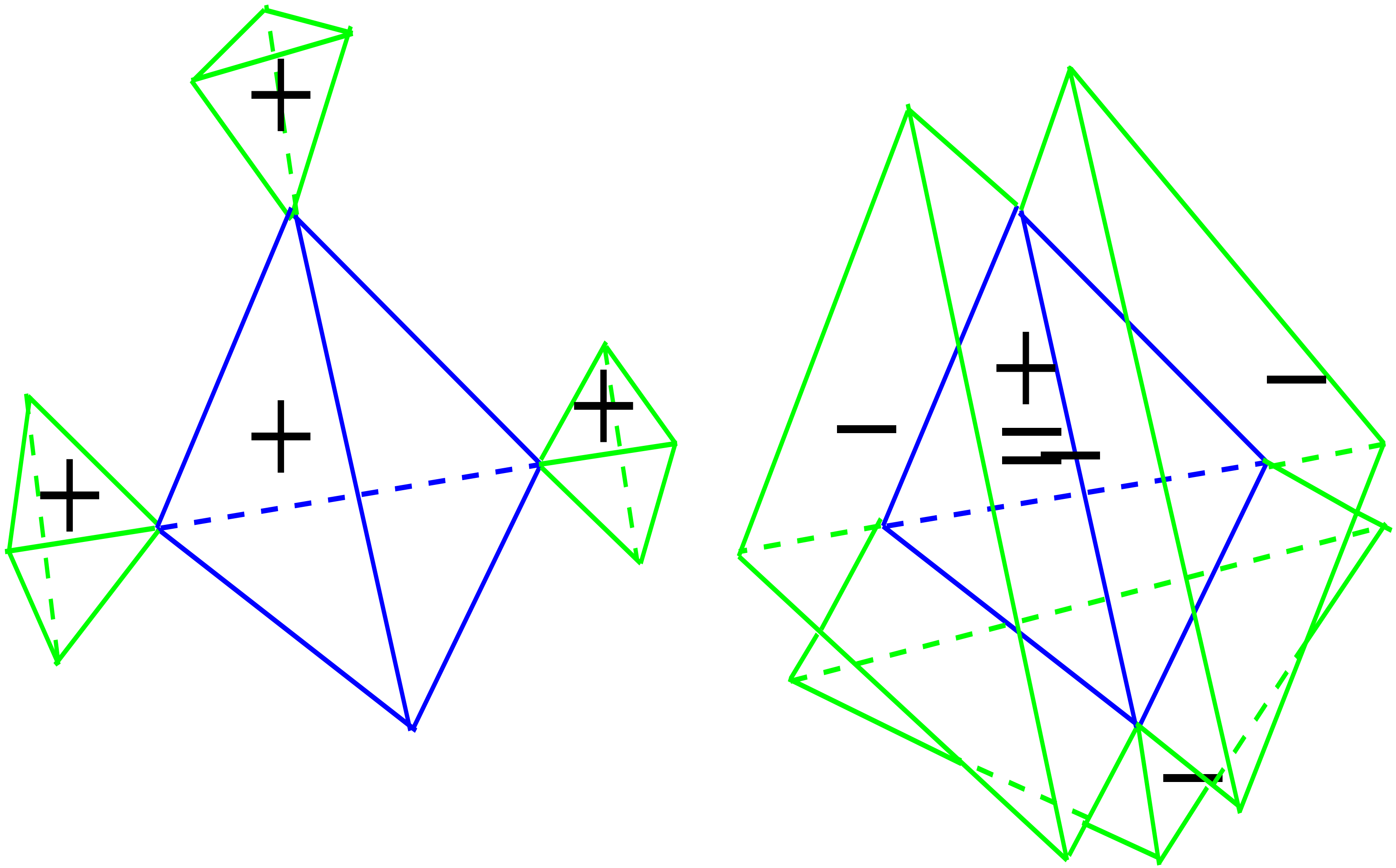
A Volume Formula

We will keep adjoining half prisms until..



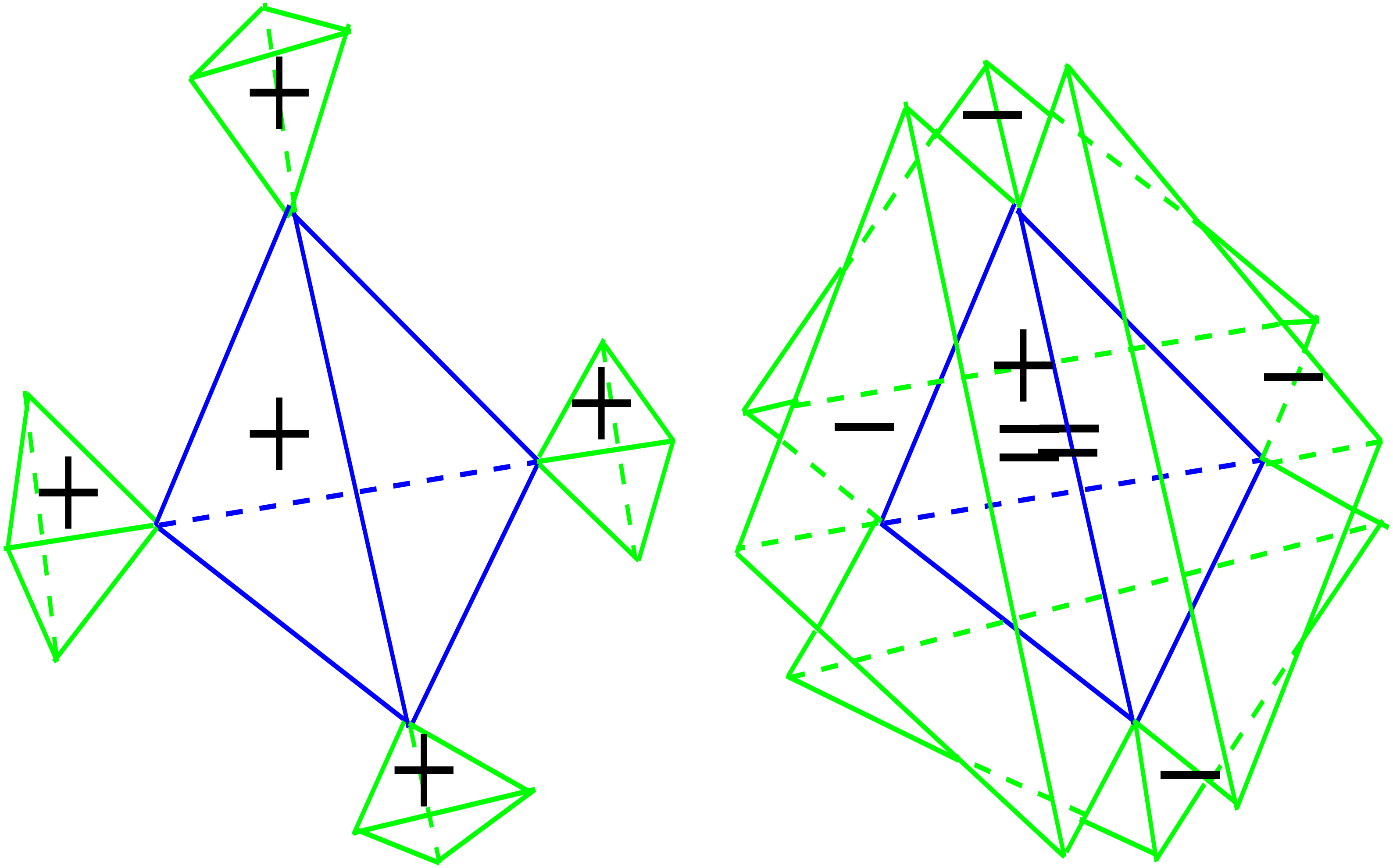
A Volume Formula

on the left we find our "usual" supertetrahedron, while on the right...



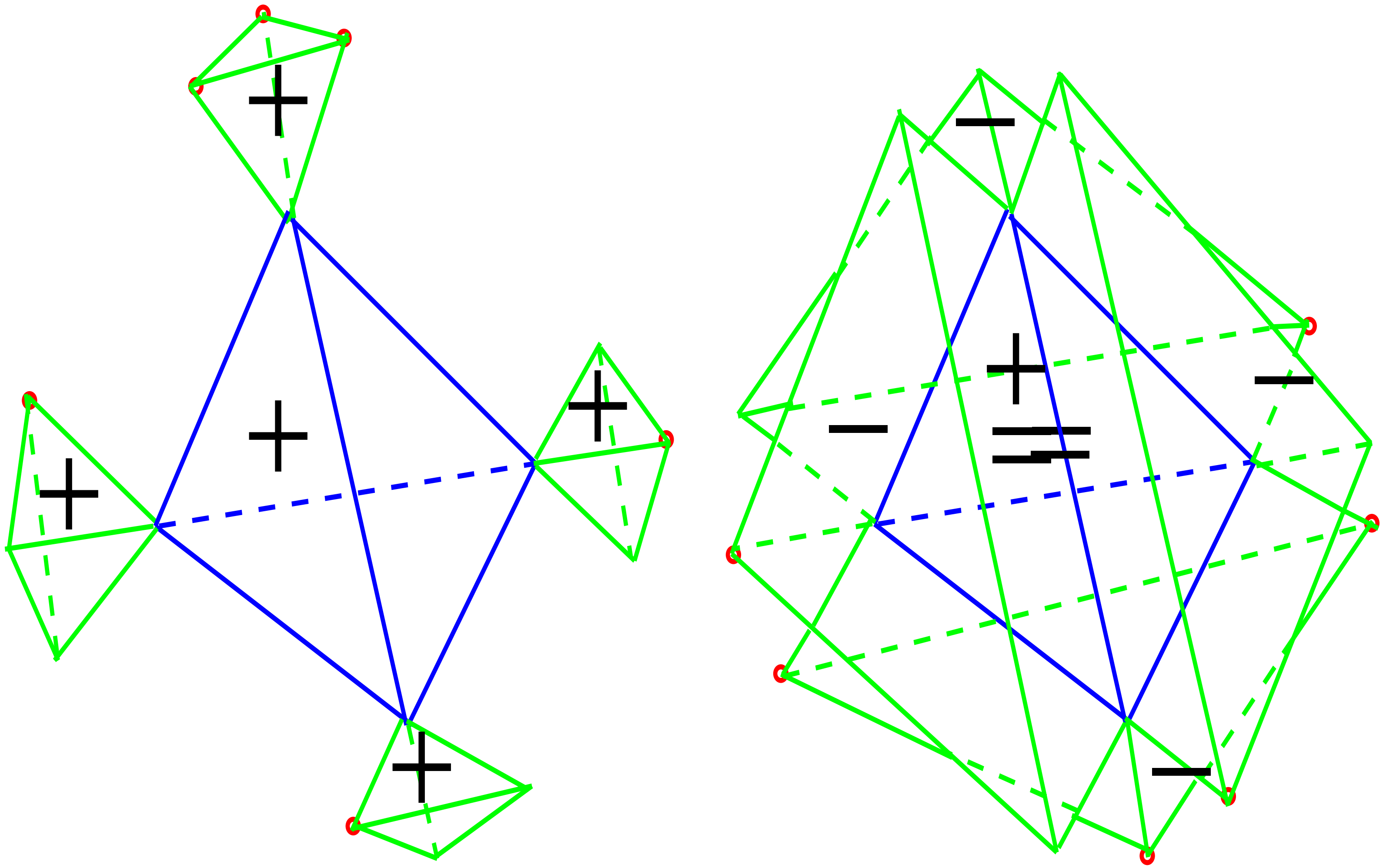
A Volume Formula

we have the supertetrahedron described by flipping all the the edges of the usual supertetrahedron. By the preservation theorem the underlying tetrahedra are the same.



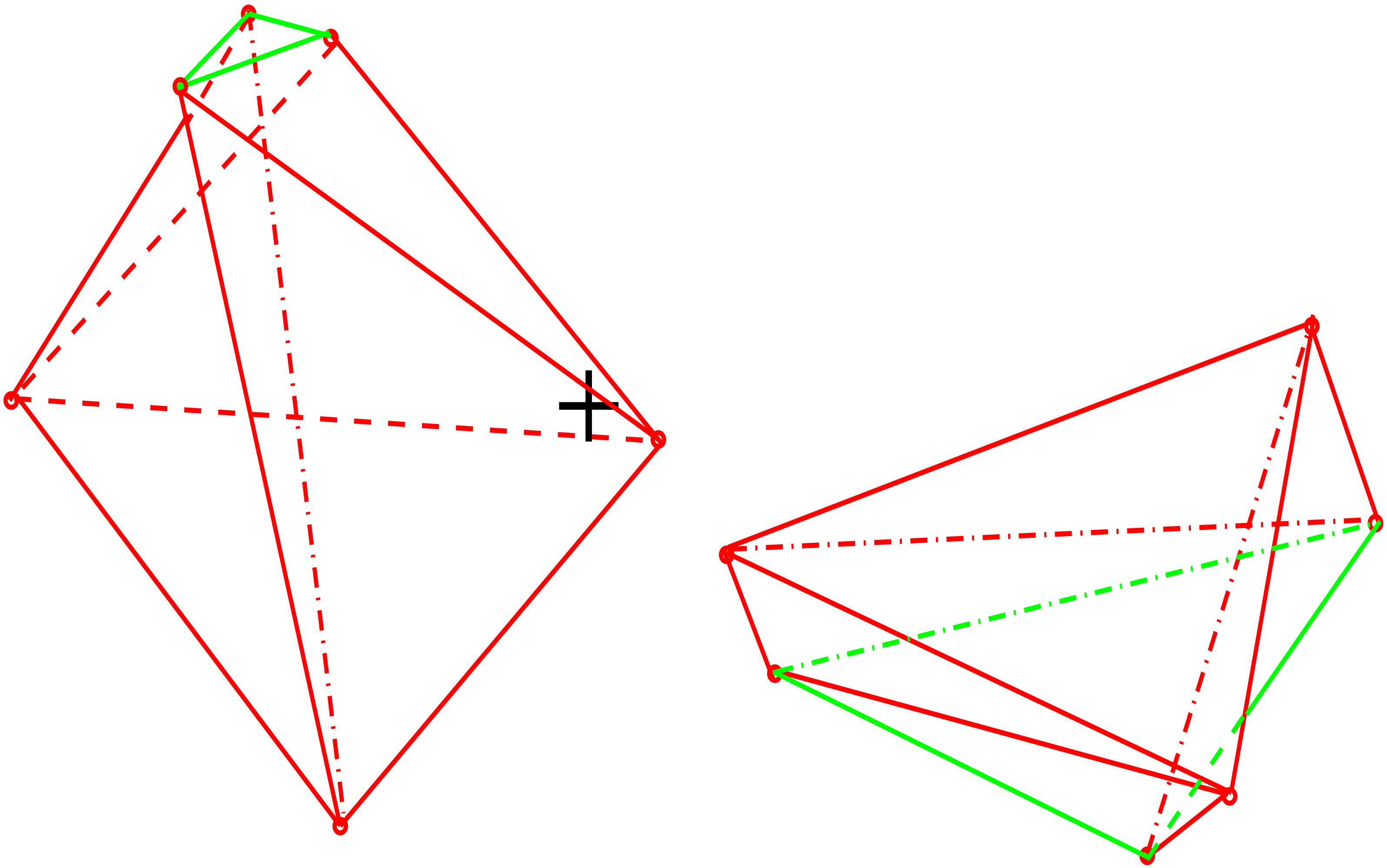
A Volume Formula

Since this was formed by edge flips all the dihedral angles are compliments. Let us cut these supertetrahedra down in the same way.



A Volume Formula

Since the angles are compliments, the ideal tetrahedra used to cut down cancel out in pairs. We have...

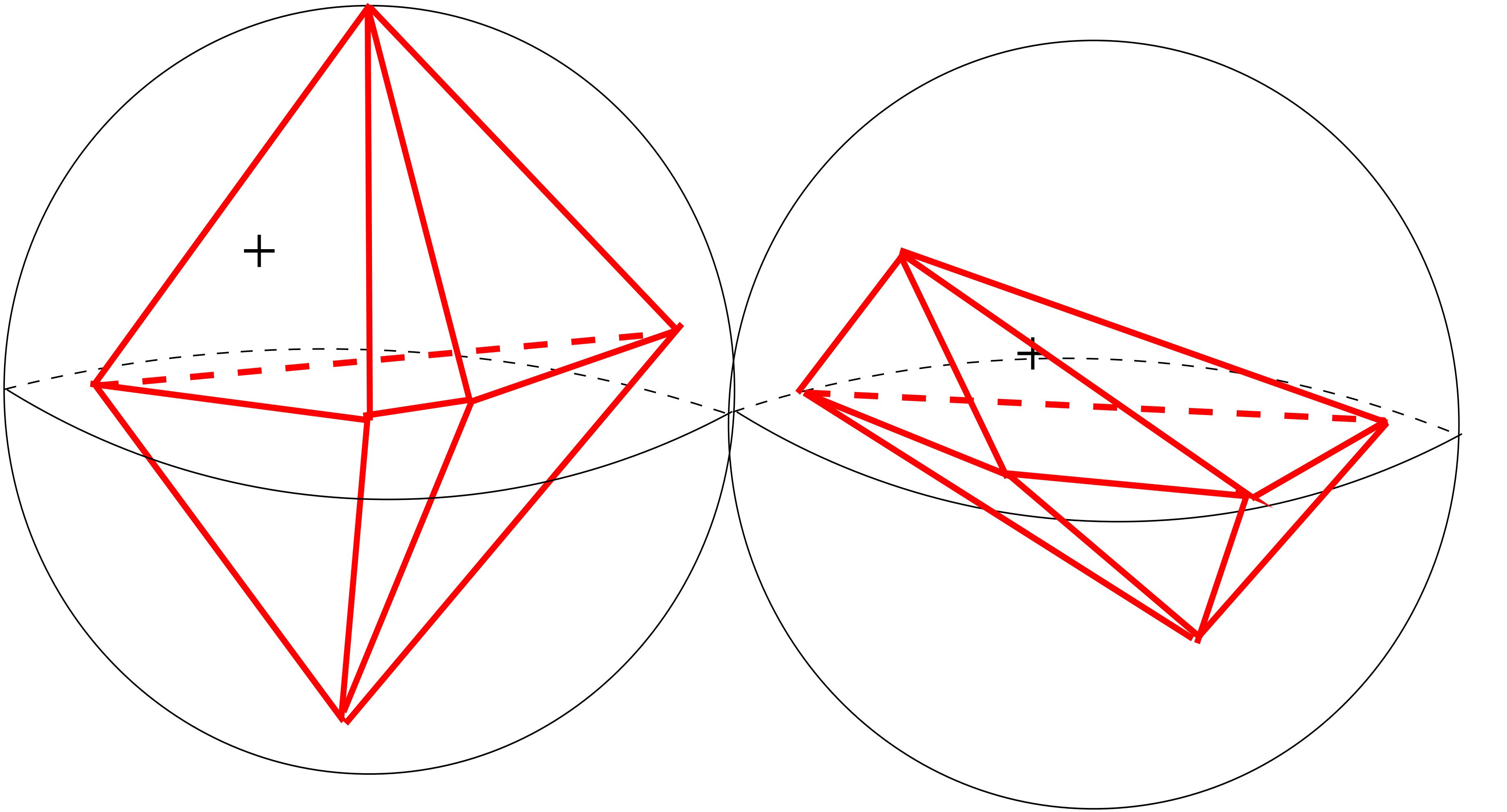


A Volume Formula

Theorem A double tetrahedron is scissors congruent to its corresponding octahedral pair.

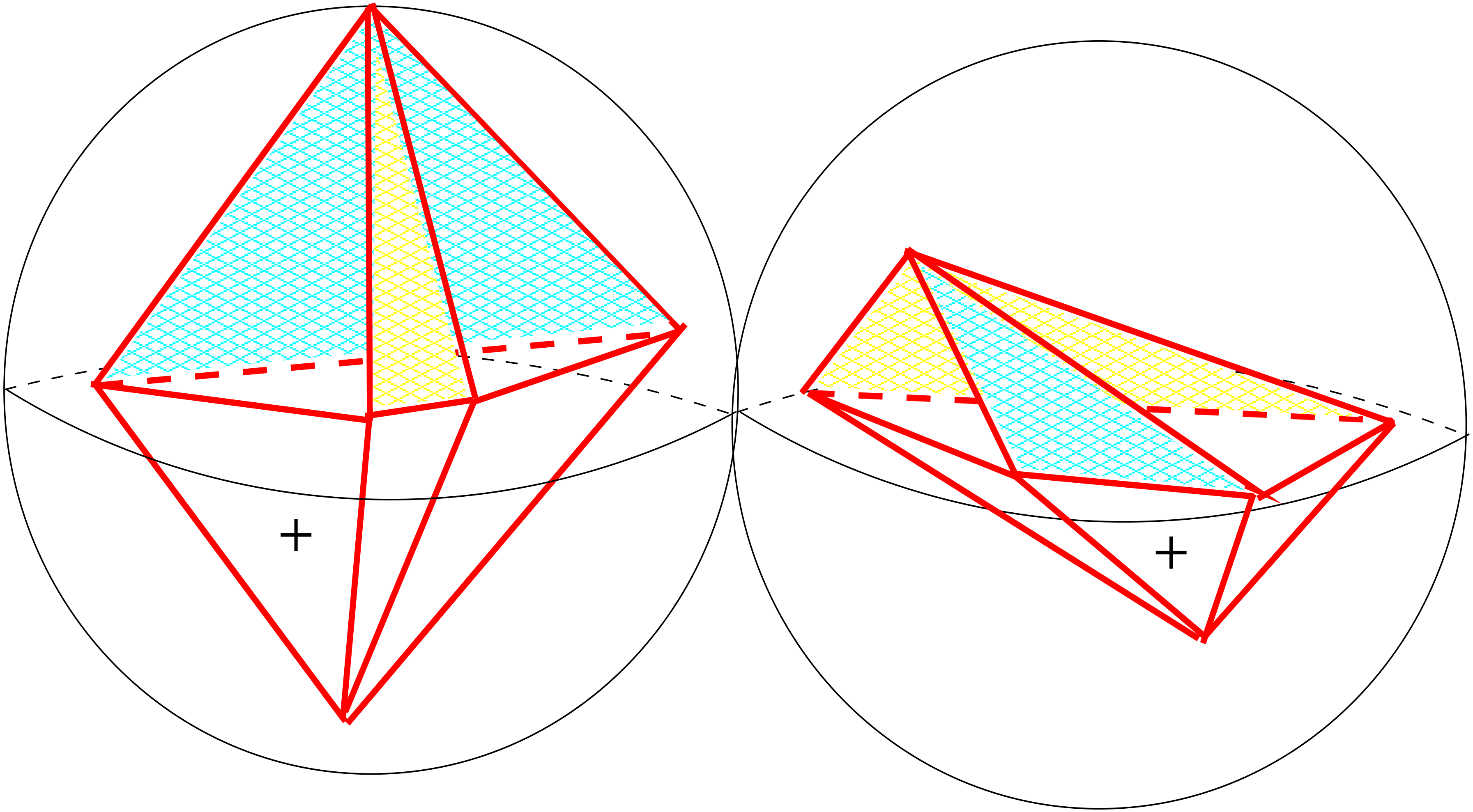
The Puff and Cut

Now we are all most there. We need to rearrange our octahedral pairs.



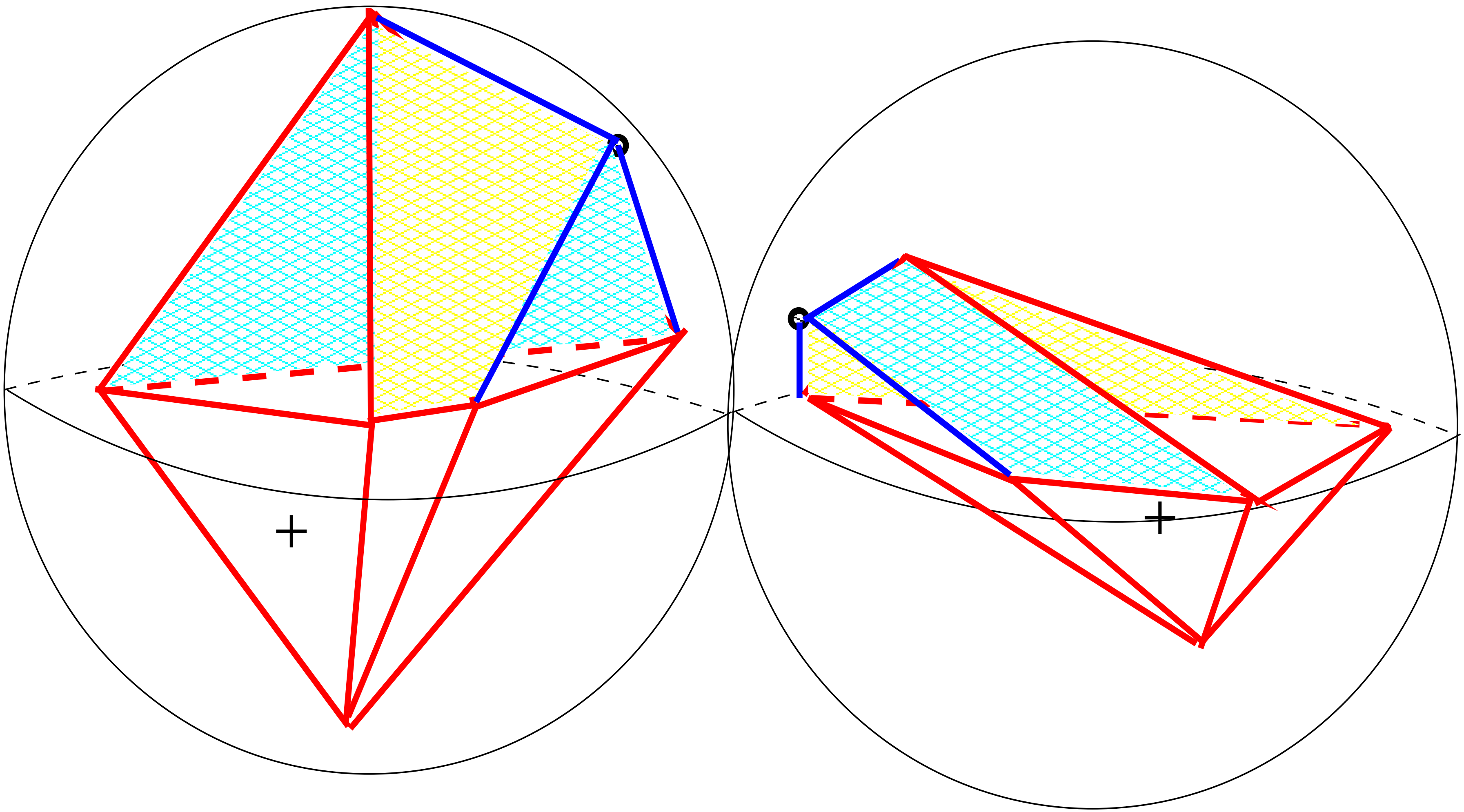
The Puff and Cut

First take a corresponding pair of planes in each octahedron.



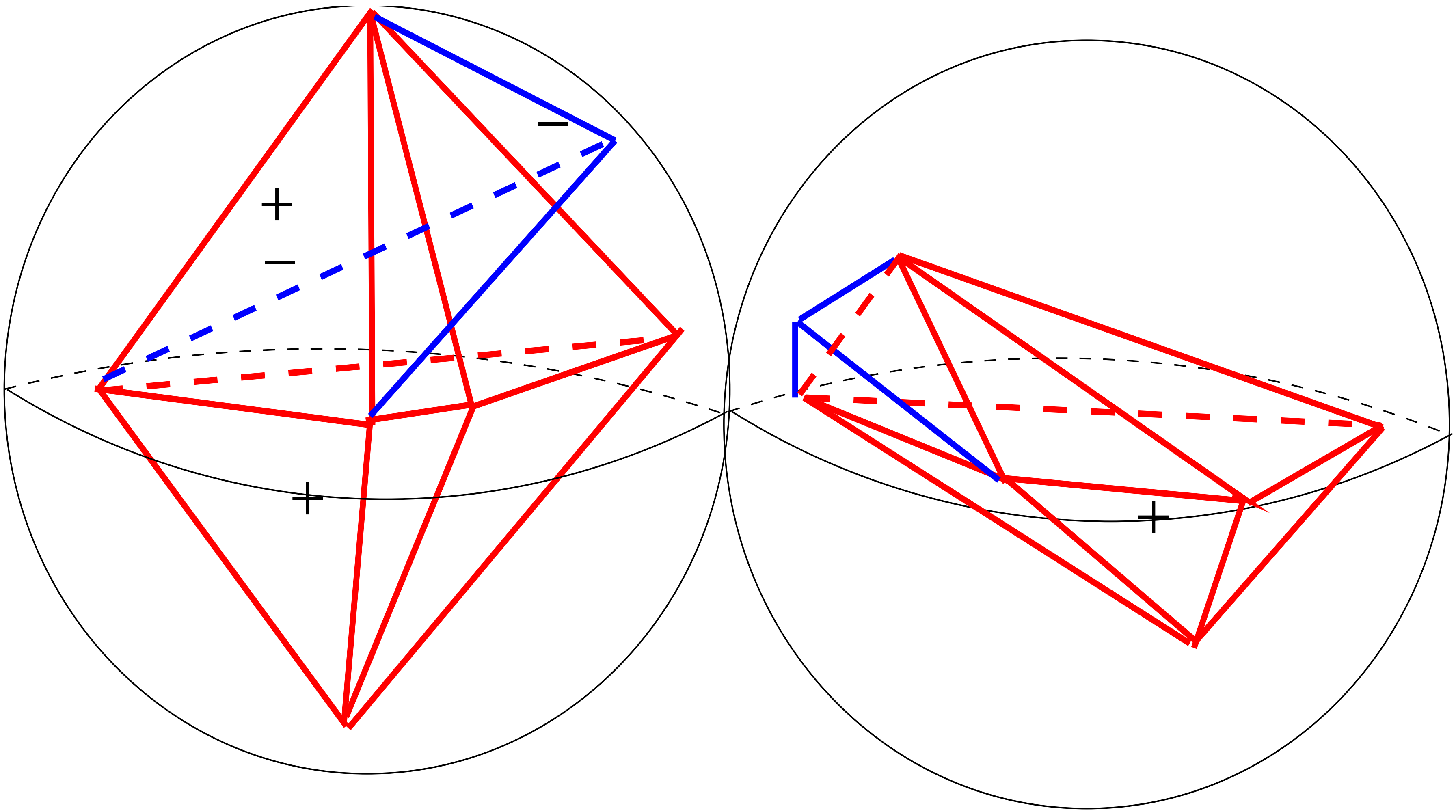
The Puff and Cut

Intersect these planes to form a new vertex associated to each octahedron.



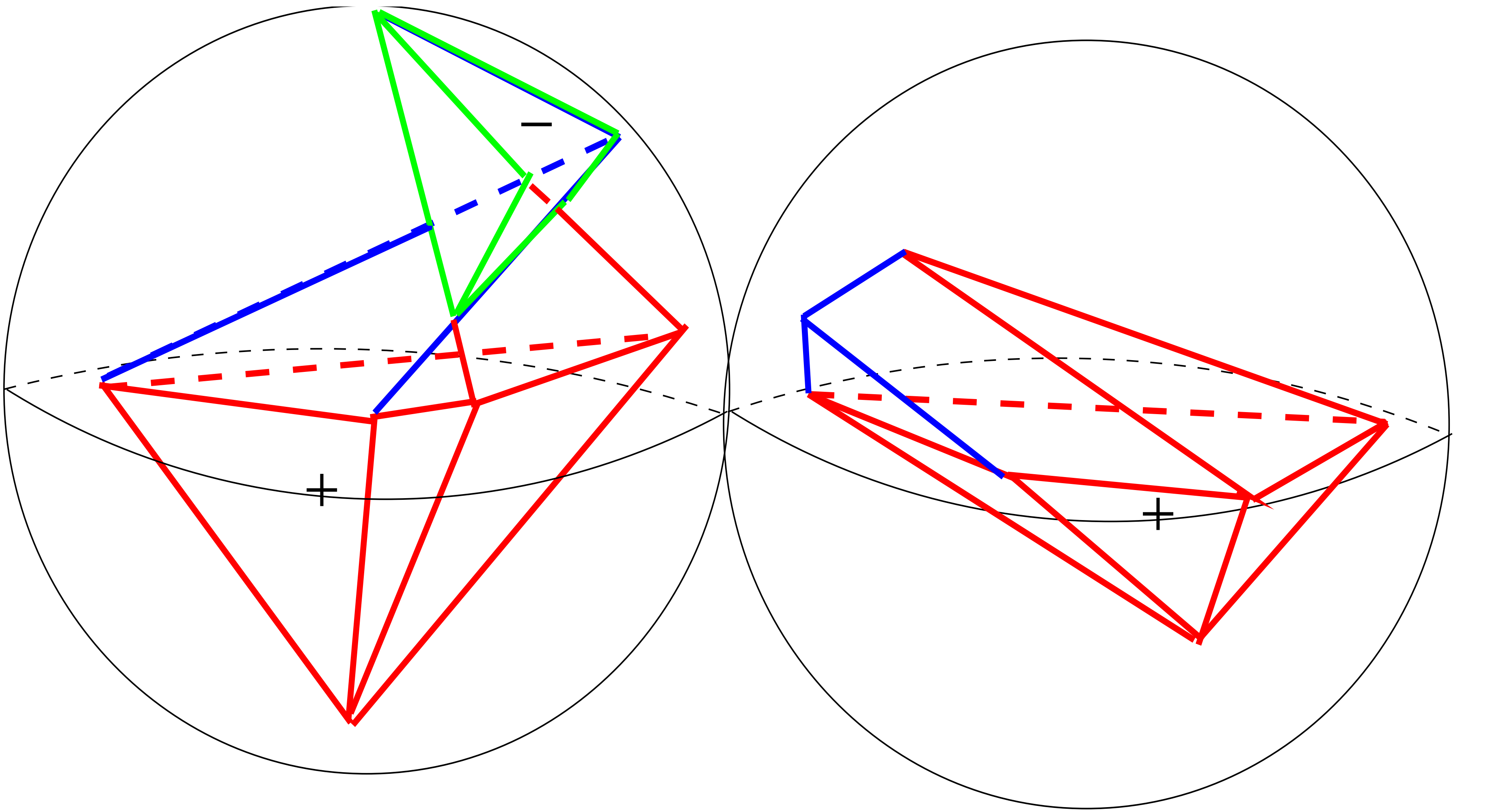
The Puff and Cut

Using this vertex we can form a pair of complimentary ideal tetrahedra.



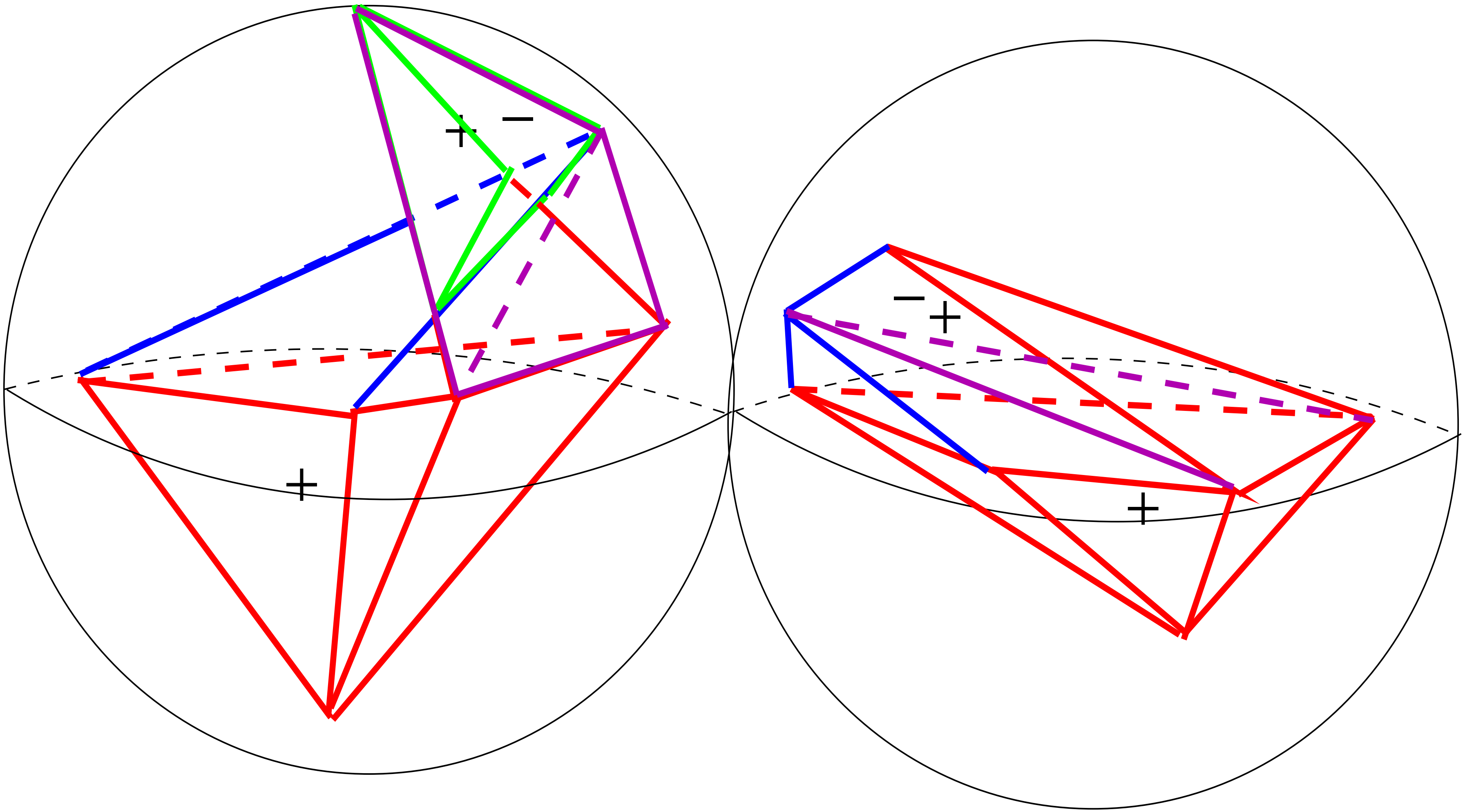
The Puff and Cut

If we "adjoin"se tetrahedra, then as a scissors class we have done nothing. This is a *puff*.



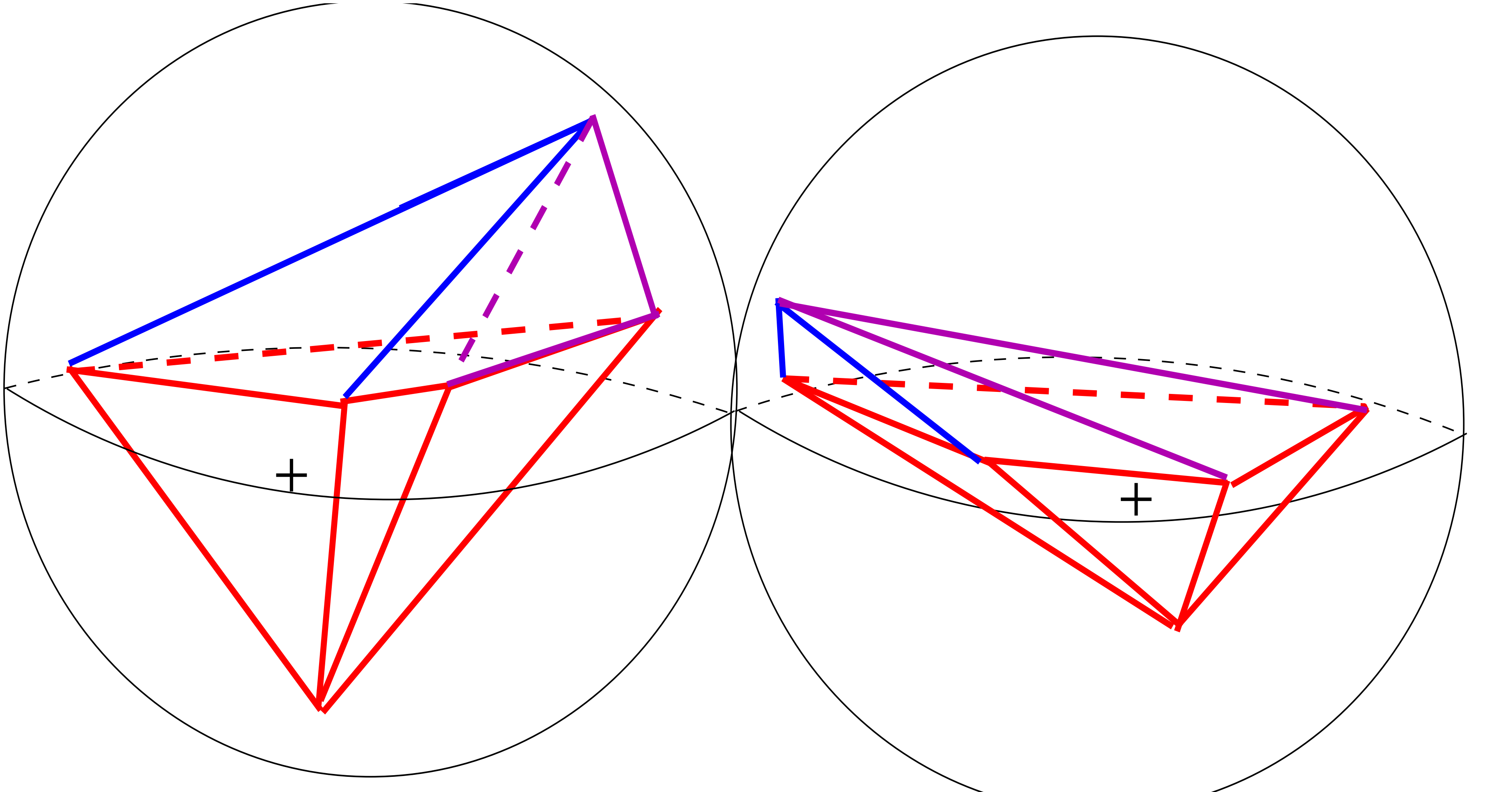
The Puff and Cut

Now the original vertex is trivalent and at our corresponding vertices we have complimentary ideal tetrahedra.



The Puff and Cut

Remove these ideal tetrahedra. As a scissors class, we still have done nothing. This is a *cut*.



Volume Symmetries

Theorem: The puff and cuts generate a group of order 23040. From each element one has a volume formula for the tetrahedron involving 8 ideal tetrahedra.

By doubling again and decomposing each of our ideal tetrahedra into Isosceles tetrahedra we have:

Theorem: The puff and cut generate a group of order 23040. From each element one has a volume formula for the tetrahedra involving 16 Isosceles ideal tetrahedra. Including the Murakami-Yano formula.

Hilbert's 13th Problem

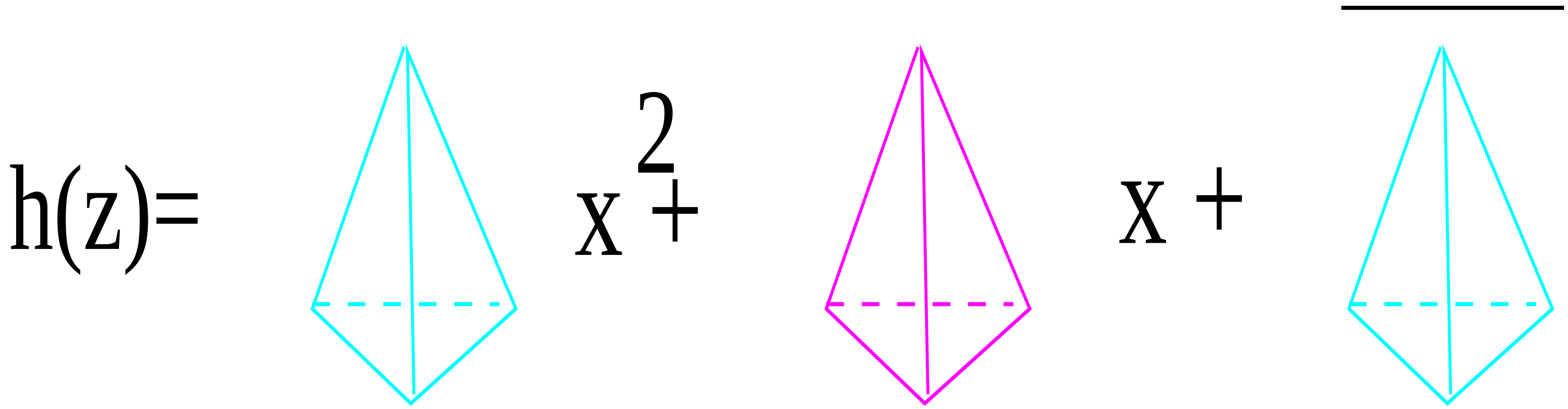
In fact we can use this sort of geometry to form other formulas. Namely we can simplify in the sense of Hilbert's 13th problem and express $F(\vec{x})$ using rational functions and as few as possible analytic functions of one variable. We will express

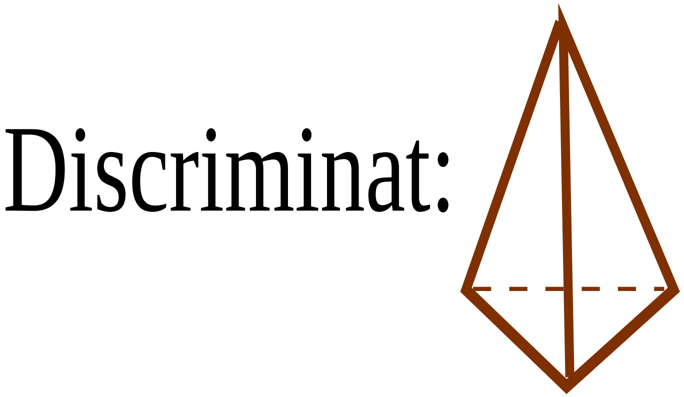
$$Vol(a,b,c,d,e,f) = \Im(\sum_{i=1}^{8} r_i F(s_i))$$

with r_i and s_i rational functions.

Hilbert’s 13th Problem

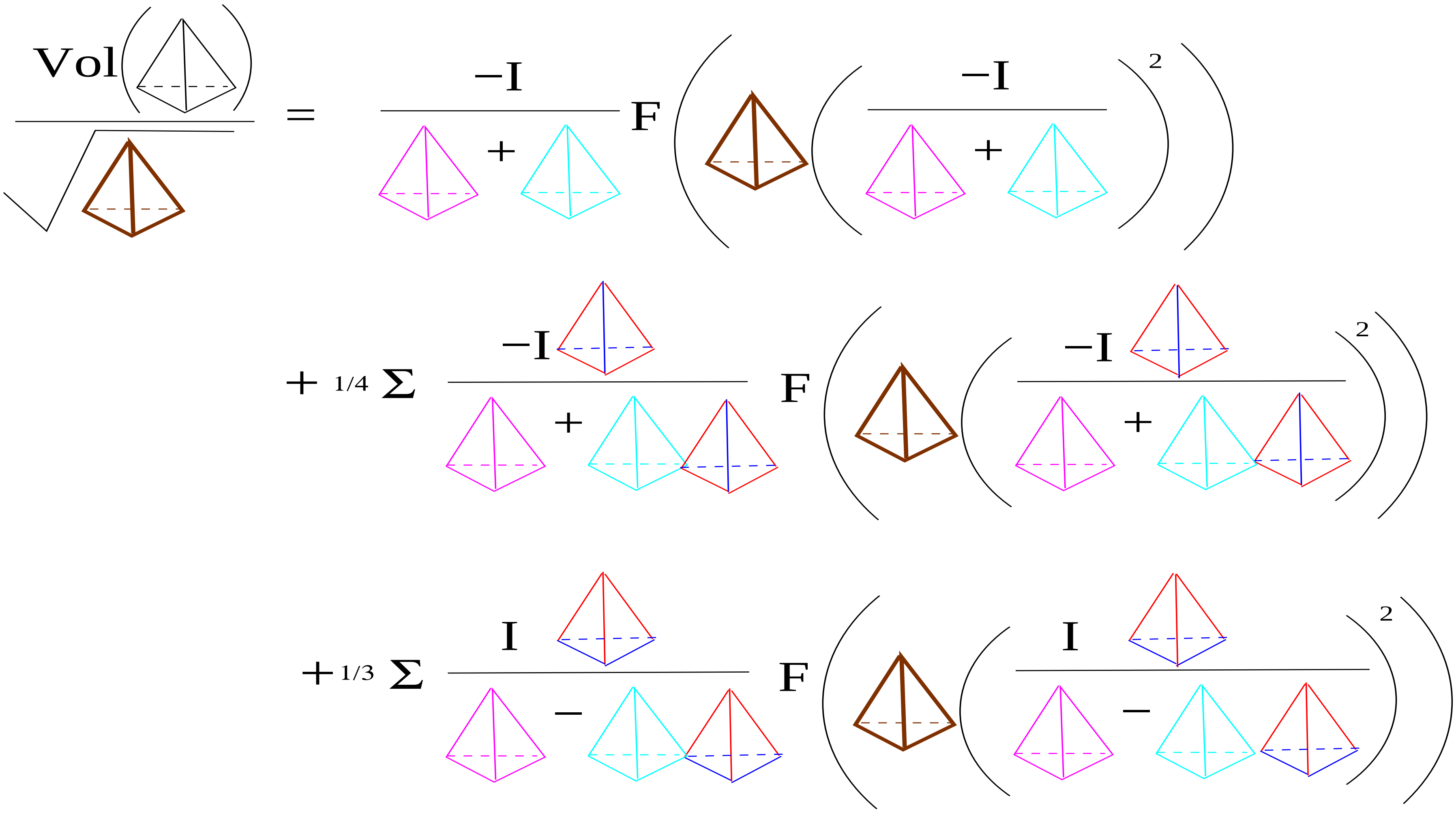
Below we name the discriminant of $h(x)$. It equals to a constant multiple of the Gram Matrix’s determinant. We are really interested in the ratio of volume and this determinant.





Hilbert's 13th Problem

Optimizing analytic simplicity we have....



Hilbert's 13th Problem

with $F(w) = \frac{H(w)}{w}$, where $H(w)$ is the odd analytic function

$$H(w) = Li_2\left(\frac{2w}{1+w}\right) + \frac{1}{4}\left(\log\left(\frac{1-w}{1+w}\right)\right)^2.$$

Corollary

Corollary:(Doyle, Leibon) There are 30 distinct tetrahedra scissors congruent to a fixed tetrahedron (generically). **Comment:** In the Euclidean case Roberts found 12, using the Regge group. Mohanty found 12 in hyperbolic case, also using the Regge group.

Corollary

The Regge group was discovered originally as an order-144 group that preserves the classical $6j$-symbol. In order to describe the Regge symmetries, let

$$S = \frac{B + C + E + F}{2}.$$

The Regge Group consist of the group generated by the tetrahedral symmetries together with the transformation

$$(A, S - B, S - C, D, S - E, S - F).$$

Regge and Pozano conjectured that from the $6j$-symbols one could reconstruct the volume of a Euclidean polyhedron, which would imply that the Regge symmetries are, in fact, Euclidean volume-preserving transformations. Roberts proved these facts.