Part II: Hyperbolic volume Higher dimensional computations and open questions

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Workshop on Hyperbolic Volumes

Waseda University, December 9 - 11, 2003

Triangulated hyperbolic 5–folds

THEOREM (A. Goncharov)

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$$\sum_{i\in I} \{z_i\} \otimes z_i = 0 \quad \text{in} \quad G(\bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}^\times$$

such that

$$\mathrm{vol}_5(M) = q \sum_{i\in I} \mathcal{L}_3(z_i) \quad \text{for some } q \in \mathbb{Q}^\times \qquad (**)$$

Here, $\{x\} \in G(F)$ where for any number field F

$$G(F) = \frac{\mathbb{Z}[\,P_1(F)\,]}{< \sum_{k=1}^{5} (-1)^k \,[\,r_2(x_1,\ldots,\widehat{x_k},\ldots,x_5)\,], [0], [\infty] \mid x_i \neq x_j >}$$

About hyperbolic sphere packings

Consider a packing $\mathcal{B}$ of r–spheres in hyperbolic space H^n. Let $B \in \mathcal{B}$ and consider its *Dirichlet–Voronoĭ–cell*

$$D(B) = \{p \in H^n \mid d(p,B) \leq d(p,B')\,,\,\forall B' \in \mathcal{B}\}$$

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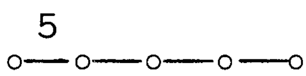
Equality holds if and only if B are the in–balls of the cells of a regular tesselation $\{m,3,\ldots,3\}$ of H^n (i.e. by regular simplices of dihedral angles π/m).

These tesselations exist precisely for $(n,m) = (2,p)$ for $p \geq 7$, $(3,6)$ and $(4,5)$ (H.S.M. Coxeter).

Problem. Prove that the simplical density function $d_n(r)$ is **strictly monotonely increasing**

Geometric application. (K, 2003)

The coset space of H^4 by the cocompact arithmetic Coxeter group



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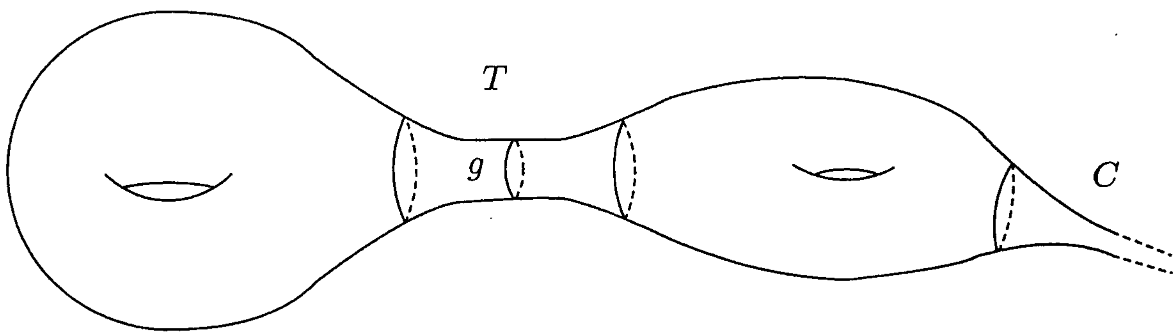

$$\exists\,\varepsilon = \varepsilon_n > 0 \;:\quad M = M_{>\varepsilon} \cup M_{\leq\varepsilon} \quad \text{with}$$

$$M_{>\varepsilon} = \{\, p \in M \mid r_p(M) > \varepsilon/2 \,\} \quad , \quad M_{\leq\varepsilon} = M \setminus M_{>\varepsilon}$$

Components of $M_{\leq\varepsilon}$

- tubes T around short simple closed geodesics g $\approx$ ball bundles over S^1
- cusp neighborhoods C $\approx$ $N^{n-1} \times \mathbb{R}_{>0}$

Canonical cusps

Let Γ be a torsion–free discrete group of hyperbolic isometries so that $\Gamma_\infty \subset \Gamma$ is non–trivial.

Let $\Lambda \subset \Gamma_\infty$ be the *translational lattice.*

By Bieberbach: Λ of finite index and of rank $n-1$

Let $\mu \neq 0$ be a *shortest* vector in Λ:

$$B_\infty(\mu) := \{\, x \in H^n \mid x_n > |\mu| \,\}$$

canonical horoball

Properties

- $B_\infty(\mu)$ precisely invariant
- $C = B_\infty(\mu)/\Gamma_\infty \subset M$ *canonical cusp*
- Canonical cusps are pairwise disjoint

Estimate of the Margulis constant

Let $n \geq 2$, $\nu = \left[\frac{n-1}{2}\right]$, and

$$c_\nu = \frac{2^{\nu+1}}{\pi^\nu} \cdot \frac{\Gamma(\frac{\nu+2}{2})^2}{\Gamma(\nu+2)} = \frac{2}{\pi^\nu} \int_0^{\pi/2} \sin^{\nu+1} t \, dt$$

THEOREM (K, 2002)

For each $\varepsilon \leq \varepsilon_n := \frac{c_\nu}{3^{\nu+1}}$, the thin part $M_{\leq \varepsilon}$ is a finite disjoint union of canonical cusps and tubes around closed geodesics of length $l \leq \varepsilon$ and of radius r satisfying

$$\cosh(2r) = \frac{1-3\kappa}{\kappa} \quad , \quad \text{where} \quad \kappa = \kappa(l) = 2\,(l/c_\nu)^{\frac{2}{\nu+1}}$$

Examples. $\varepsilon_2 = 2/3 \simeq 0.6666$, $\varepsilon_3 = 1/18 \simeq 0.0555$
$\varepsilon_5 \simeq 0.0050$

Compare

- $\varepsilon_2 \geq \text{arsinh}(1) \simeq 0.8813$ (P. Buser, 1978)
- $\varepsilon_3 \geq 0.104$ (R. Meyerhoff, 1987)
- $\varepsilon_n \geq \sqrt{3}/9\pi \simeq 0.0612$ for $n = 4, 5$ (K, 2001, 2002)

Applications

- **Volume bound :** $\exists p \in M$ such that

$i_p(M) > \frac{1}{(n+3)\,\pi^{n-1}}$ and $\mathrm{vol}_n(M) > \frac{\Omega_{n-1}}{n} \frac{1}{[(n+3)\pi^{n-1}]^n}$

Remark. If M is non-compact with $m \geq 1$ cusps, then

$$\mathrm{vol}_n(M) \geq m \frac{2^n}{n(n+1)} \mathrm{vol}_n(S^{\infty}_{reg})$$

- For M is compact,

$$i(M) \geq \frac{c(n)}{(\sinh(\mathrm{diam}\,(M)))^{[\frac{n+1}{2}]}}$$

- **Gromov's invariant:** For $n = 2, 3$ and $n >> 1$,

$||M|| > \gamma_n$, where $\gamma_n := \frac{\Omega_{n-1}}{8n^2} \frac{1}{\left[(n+1)\pi^{n-1}\right]^n}$

- **Number of manifolds with volume bound:]**

In the estimate $V^{\alpha V} \geq \rho_n(V) \geq V^{\beta V}$ of [BGLM],

$$\log \rho_n(V) < \frac{n^2 \exp(\,23\,n(n+1))}{\Omega_{n-1}} V \log V$$

Compare with Gromov's estimate

$$\rho_n(V) \leq V \exp(\exp(\exp(n+V)))$$

Mahler measures

Let $P(x_1, \ldots, x_n)$ be an $n-$variable Laurent polynomial over $\mathbb{C}$.

$$m(P) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |\, P(e^{i\theta_1}, \ldots, e^{i\theta_n}) \,|\, d\theta_1 \cdots d\theta_n$$

(logarithmic) Mahler measure of P

For $P(x) = a \prod_j (x - \alpha_j)$, Jensen's formula yields

$$m(P) = \log |a| + \sum_j \log_+ |\, \alpha_j \,|$$

where $\log_+ x = \max(0, \log x)$ as usually

For $n > 1$, some Mahler measures are related to **polylogarithms**

Mahler measures for few variables

The exemplary results of Chris Smyth (1981)

- $$\pi\, m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\, L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) \frac{1}{n^s}$$
$$= 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \cdots$$

Consider the 3–variable polynomial

$P_{a,b,c}(x,y,z) := a + bx^{-1} + cy + (a+bx+cy)z \ , \ a,b,c \in \mathbb{R}$:

- $\pi^2\, m(1+x+y+z) = 7\zeta(3)/2$
- $\pi^2\, m(1+x^{-1}+y+(1+x+y)z) = 14\zeta(3)/3$

Hints for the proof

- ∘ Relate $m(P_{a,b,c})$ to $m(P_{0,c,1})$ and $m(P_{1,b,c})$
- ∘ $\pi^2\, m(P_{0,c,1}) \quad \longleftrightarrow \quad -\mathrm{Re} \int\int \mathrm{Log}(x+y) \frac{dx}{x} \frac{dy}{y}$
- ∘ $\pi^2\, m(P_{1,b,c}) \quad \longleftrightarrow \quad -\mathrm{Re} \int\int \mathrm{Log}(1+x+y) \frac{dx}{x} \frac{dy}{y}$
- ∘ For $n = 2, 3$, compare with

$$\Lambda_n(z) = \frac{1}{(n-1)!} \int_0^z \frac{(-\mathrm{Log}\, t)^{n-1}}{1-t}\, dt = \sum_{r=0}^{n-1} \frac{(-\mathrm{Log}\, z)^r}{r!} \mathrm{Li}_{n-r}(z)$$

An n–variable Mahler measure

Theorem (Smyth, preprint 2003)

Let $n \geq 3$. Then,

$$m((x_1 + x_1^{-1}) \cdots (x_{n-2} + x_{n-2}^{-1}) + 2^{n-3}(x_{n-1} + x_n)) =$$

$$(n-3)\log 2 + \left(\tfrac{2}{\pi}\right)^{n-1} {}_{n+1}F_n\left(\{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1, \ldots, 1\}, \{\tfrac{3}{2}, \ldots, \tfrac{3}{2}\}, 1\right)$$

Here, the **generalised hypergeometric function** ${}_rF_m$ is defined by

$${}_rF_m(\{a_1, \ldots, a_r\}, \{b_1, \ldots, b_m\}, z) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k\, z^k}{(b_1)_k \cdots (b_m)_k\, k!}$$

$$\text{with} \quad (a)_k := a(a+1)\cdots(a+k-1)$$

Corollary.

$$m(1 + x + y + z) = \tfrac{4}{\pi^2}\, {}_4F_3(\{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1\}, \{\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\}, 1)$$

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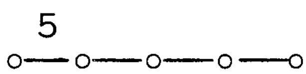

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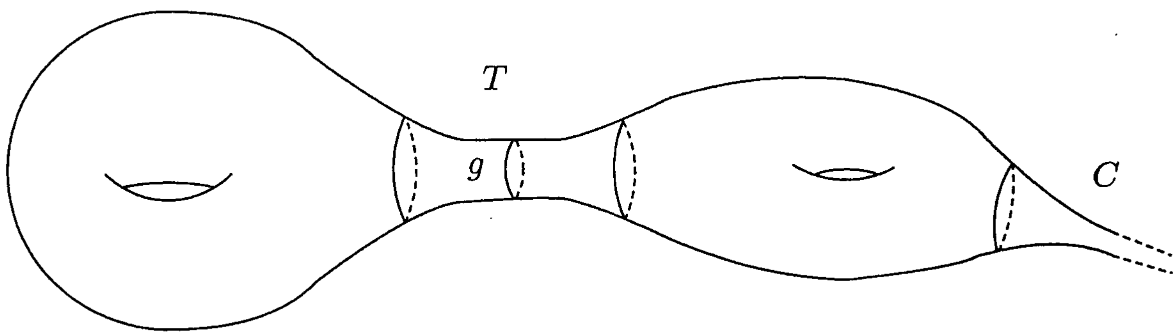

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