Part I: Non–euclidean volume Introduction and survey

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Relations to characteristic invariants

F. Hirzebruch, 1991: *"Kombinatorik in Geometrie"*

Euler characteristic, Betti numbers, signature, ...
of certain algebraic and topological varieties are related to tangent numbers and Euler numbers

- By previous result, the Euler-Poincaré characteristic of a triangulated manifold of even dimension is expressible in terms of tangent numbers

- Let A be a principally polarised $n-$dimensional abelian variety and D a theta divisor associated to the polarisation.

D is of complex dimension $n-1$ and in general smooth. Its **holomorphic Euler-characteristic** $\chi^k(D)$ with coefficients in the sheaf of germs of holomorphic $k-$forms for $1 \leq k \leq n-1$ can be computed by means of the **Theorem of Riemann–Roch–Hirzebruch** and identified with Euler numbers

The **signature** of the $(2n-2)-$dimensional smooth manifold D is expressible in terms of tangent numbers

Through the action of A on itself arise r theta-divisors in general position with intersection $D^{(r)}$. This is a smooth variety of dimension $n-r$ whose **arithmetic genus** is related to the Stirling number S_r^n, i.e. to the number of partitions of a set with n elements into r disjoint non-empty subsets

Further results in even dimensions

- K, 1991 :
  Reduction formulas for d–truncated orthoschemes with $0 \leq d \leq 2$ in H^{2n}

- Thomas Zehrt, PhD thesis, 2003 :
  Various reduction formulas and applications *without* use of Schläfli's differential formula of the type

$$\frac{2\,K^n}{\mathrm{vol}_{2n}(S^{2n})}\,\mathrm{vol}_{2n}(P) = \sum_{i=0}^{n} \sigma_j(F^{2j})\;\alpha_{2n-2i-1}(F^{2j})$$

Example For cubes $W \subset X_K^{2n}$

$$\frac{2\,K^n}{\mathrm{vol}_{2n}(S^{2n})}\,\mathrm{vol}_{2n}(W) = \sum_{i=0}^{n} (-1)^i E_{2i}\;\omega_{2n-2i-1}(W)$$

Euler numbers $E_{2i} = 1, 1, 5, 61, 1385, 50521, \ldots$

$$\frac{1}{\cos(Kz)} = \sum_{i=0}^{\infty} K^i\,\frac{E_{2i}}{(2i)!}\;z^{2i}$$

$$E_0 = 1 \quad , \quad (-1)^i\,E_{2i} = \sum_{v=0}^{i-1} (-1)^{v+1}\binom{2n}{2v}\,E_{2v}$$

Rational Simplex Conjecture of Schläfli

(Cheeger-Simons)

Let $n \geq 3$. The following list contains ALL simplices in S^n such that

- angles in $]0, \pi]$ are commensurable with π
- volume is commensurable with $\mathrm{vol}_n(S^n) = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$

TYPE	VOLUME/ $\mathrm{vol}_n(S^n)$
n=3	
o———o———o——o	1/60
o———o———o—4—o	1/192
o———o—4—o——o	1/576
o———o———o—5—o	7200
o———o———o—5/2—o	191/7200
o—5/2—o———o—5—o	1/360
o—5/2—o—5—o——o	1/1800
o—5—o—5/2—o——o	19/1800
o—5—o—5/2—o—5—o	1/1200
o—5/2—o—5—o—5/2—o	11/1200
o—5/3—o—5—o—5/2—o	49/1200
o—5/4—o—5/2—o——o	191/1800
o—5/2—o—5/4—o—5/2—o	191/1200
n>3	
o———o——— ··· ———o——o	1/(n+1)!
o———o——— ··· ———o—4—o	$1/2^{n+1}$(n+1)!

Volumes in even dimensions
Reduction formulas

In X_K^n of curvature $K = \pm 1$, normalise volume

$$f_n = \frac{2^{n+1}}{\mathrm{vol}_n(S^n)}\,\mathrm{vol}_n \quad , \quad f_0 := 1$$

THEOREM (Schläfli's Reduction formula)

Let $S \subset X_K^{2n}$ be a simplex with scheme Σ. Then,

$$K^{n/2}\, f_{2n}(\Sigma) = \sum_{k=0}^{n} (-1)^k a_k \sum_{\sigma} f_{2n-(2k+1)}(\sigma) \;,\; \sum f_{-1} := 1 \quad ,$$

where σ runs through all elliptic subschemes of order $2(n-k)$, and

$$\tan x = \sum_{k=0}^{\infty} \frac{a_k}{(2k+1)!}\, x^{2k+1}$$

Tangent numbers a_k are expressible in form of

$$a_k = 2^{2k+1}\,\frac{2^{2k+2}-1}{k+1}\, B_k = 1, 2, 16, 16{\cdot}17, 256{\cdot}31, \ldots\ldots$$

Application

On S^3, there is the period of *orthoschemes* given by

$$\alpha\,,\ \beta\,,\ \gamma\,,\ \pi/2 - a\,,\ b\,,\ \pi/2 - c$$

allowing to compute, e.g., the volumes of all spherical Coxeter simplices

The many aspects of hyperbolic volume

dilogarithms

Dedekind $\zeta_F(2)$

Hilbert's 3rd Problem

Borel regulator

hyperbolic 3–volume

Milnor's Conjecture for Л_2

Mahler measure in 2 variables

central charges of RCFTs

link invariants, knot complements and Kashaev's Conjecture

....

Triangulated hyperbolic 3–folds

THEOREM (Thurston, Neumann–Zagier,...)

Let M be an oriented hyperbolic 3–manifold of finite volume. Then, there are finitely many $z_i \in \bar{\mathbb{Q}}$, $i \in I$, satisfying

$$\sum_{i \in I} z_i \wedge (1 - z_i) = 0 \quad \text{in} \quad \bigwedge{}^2 \bar{\mathbb{Q}}^\times \qquad (*)$$

such that

$$\mathrm{vol}_3(M) = \sum_{i \in I} \mathcal{L}_2(z_i)$$

Dehn invariant of a polyhedron

For $P \subset H^3$ with edges of length l and attached dihedral angle α_l :

$$\Delta(P) = \sum_{l\, edge\, of\, P} l \otimes \alpha_l \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$$

extendable to asymptotic polyhedra (cut off ideal vertices by means of small horospheres and measure then length of truncated edge)

Example

$$\begin{aligned}\Delta(S_\infty(z)) &= 2\,\{\,\log|1-z| \otimes \arg z - \log|z| \otimes \arg(1-z)\} \\ &= z \wedge (1-z) - \bar{z} \wedge (1-\bar{z}) \qquad \text{where}\end{aligned}$$

$$\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z} \cong \bigwedge{}^2 (\mathbb{C}^\times)^- \;:\; r \otimes \theta \bmod 2\pi \;\mapsto\; -e^r \wedge e^{i\theta}$$

Kashaev's volume conjecture

R. Kashaev: Hyperbolic 3–volume of link complements of S^3 should equal classical limits of Kashaev's link invariant defined by quantum dilogarithms (true for the knots 4_1, 5_2, 6_1)

H. Murakami–J. Murakami: Kashaev's invariant is a specialisation of the colored Jones function evaluated at special values

Example For the Figure Eight–knot 4_1

the *N-th colored Jones polynomial* equals (Habiro–Le)

$$J_N(4_1;t) = \sum_{k=0}^{N-1} \prod_{j=1}^{k} \left(t^{(N+j)/2} - t^{-(N+j)/2}\right)\left(t^{(N-j)/2} - t^{-(N-j)/2}\right)$$

and has the **asymptotic behavior** for fixed $r \in \mathbb{N}$

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(4_1; \exp(2\pi i r/N)|}{N} =$$

$$= \frac{2}{r}\left\{ \text{Л}_2(r\pi + \theta(r)/2) - \text{Л}_2(r\pi - \theta(r)/2) \right\} = \frac{\mathrm{vol}_3(S^3 \setminus 4_1)}{r} \quad ,$$

with $\theta(r) \in \mathbb{N}$ satisfying $\cos\theta(r) = \cos(2\pi r) - 1/2$

r=1: R. Kashaev ; r>1: H. Murakami

Some arithmetic link complements

- Borromean ring complement $B = S^3 \setminus 6^3_2$

$$\mathrm{vol}_3(B) = 2\,\mathrm{vol}_3(O^{\infty}_{reg}(\frac{\pi}{2})) = 16\,Л(\frac{\pi}{4}) \simeq 7.3276$$

- Whitehead link complement $W = S^3 \setminus 5^2_1$

$$\mathrm{vol}_3(W) = \mathrm{vol}_3(O^{\infty}_{reg}(\frac{\pi}{2})) = 8\,Л(\frac{\pi}{4}) \simeq 3.6638$$

- Figure Eight knot complement $E = S^3 \setminus 4_1$

$$\mathrm{vol}_3(E) = 2\,\mathrm{vol}_3(S^{\infty}_{reg}(\frac{\pi}{3})) = 6\,Л(\frac{\pi}{3}) \simeq 2.0298$$
$$= \frac{\sqrt{3}}{\pi^2}\,\zeta_{\mathbb{Q}(\sqrt{-3})}(2) \quad , \quad \text{where}$$

in general, for an imaginary quadratic number field $F = \mathbb{Q}(\sqrt{-d})$, $d \geq 1$ squarefree, with ring of integers $\mathcal{O}_d$, the covolume of the discrete subgroup $PSL(2, \mathcal{O}_d)$ of $PSL(2, \mathbb{C})$ is given by **Humbert's formula**

$$\mathrm{vol}_3(H^3/PSL(2, \mathcal{O}_d)) = \frac{d^{3/2}}{4\pi^2}\,\zeta_{\mathbb{Q}(\sqrt{-d})}(2) = \frac{d^{3/2}}{24}\sum_{r=1}^{\infty}\left(\frac{-d}{r}\right)\frac{1}{r^2}$$

C. Smyth, 1981:

$$2\pi m(1 + x + y) = \frac{3\sqrt{3}}{2}\sum_{r=1}^{\infty}\left(\frac{-3}{r}\right)\frac{1}{r^2} = \mathrm{vol}_3(E)$$

Generalised 3rd Problem of Hilbert

Problem : $P_1, P_2 \subset X^3_K$ scissors congruent

if and only if

$$\mathrm{vol}_3(P_1) = \mathrm{vol}_3(P_2)\ ,\ \mathrm{Dehn}(P_1) = \mathrm{Dehn}(P_2)$$

where

$$\mathrm{Dehn(P)} = \sum_F l(F) \otimes \alpha_F \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$$

THEOREM (J. Dupont–C.H. Sah)

Consider $S_\infty(\frac{1}{n}) = (\infty, 0, 1, \exp(2\pi i/n)) \subset \overline{H^3}$, $n \geq 7$.

Let $\theta \in]\frac{1}{6}, \frac{1}{2}[$ so that $\text{Л}_2(\theta\pi) = \text{Л}_2(\pi/n) = \frac{1}{2}\mathrm{vol}_3(S_\infty(\frac{1}{n}))$.

Then, there is the following alternative :

(1) $\theta \in \mathbb{R} - \mathbb{Q}$, i.e. $\mathrm{Dehn}(S_\infty(\theta)) \neq 0$, and hence there is a pair of hyperbolic tetrahedra with *equal volume* and *different Dehn-invariant*

(2) $\theta \in \mathbb{Q}$, i.e. Milnor's Conjecture is *false*

Milnor's Conjecture

Let $\{\theta_j\} \subset \mathbb{Q}\pi$:

Every $\mathbb{Q}$–linear relation

$$\sum_j q_j \text{Л}_2(\theta_j) = 0 \quad \text{with} \quad q_j \in \mathbb{Q}$$

is a consequence of the relations

$$\text{Л}_2(x+\pi) = \text{Л}_2(x) \quad , \quad \text{Л}_2(-x) = \text{Л}_2(x)$$

$$\text{Л}_2(nx) = n \sum_{k \bmod n} \text{Л}_2(x + \frac{k\pi}{n})$$

distribution law

The formula of Cho–Kim

Let T be a hyperbolic tetrahedron with dihedral angles $A, B, C\,;\ D, E, F$.

For solutions $(P_1, Q_1, R_1, S_1, T_1)$, $(P_2, Q_2, R_2, S_2, T_2)$ of

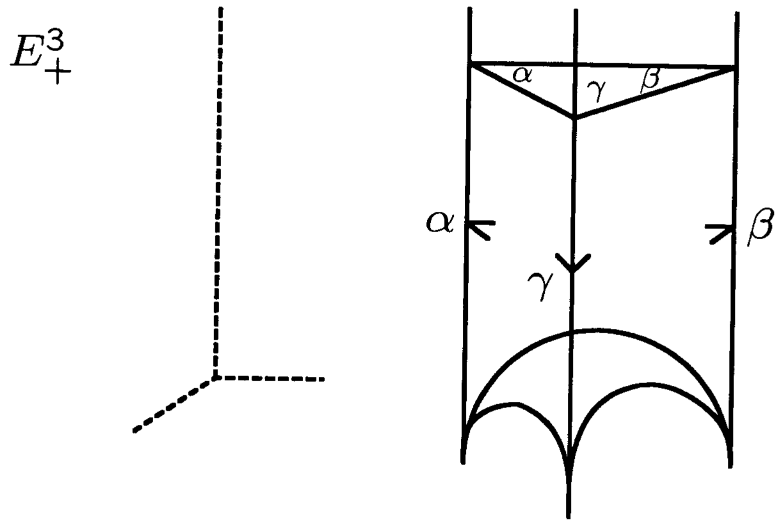
$P{+}Q = B\,,\ R{+}S = E\,,\ Q{+}R{+}T = F{+}\pi\,,\ P{+}S{+}T = C{+}\pi\,,$

$$\begin{vmatrix} 1 & -\cos D & -\cos P & \cos B & \cos C \\ -\cos D & 1 & \cos(R+T) & \cos F & \cos E \\ -\cos P & \cos(R+T) & 1 & -\cos Q & \cos(S+T) \\ \cos B & \cos F & -\cos Q & 1 & -\cos A \\ \cos C & \cos E & \cos(S+T) & -\cos A & 1 \end{vmatrix} = 0$$

$$\begin{aligned} 2\,\mathrm{vol}_3(T) = {} & Л_2(P_1) - Л_2(Q_1) + Л_2(R_1) - Л_2(S_1) - \\ & - Л_2\left(\tfrac{B-C-A+\pi}{2} - Q_1\right) + Л_2\left(\tfrac{D-B-F+\pi}{2} + Q_1\right) + \\ & + Л_2\left(\tfrac{E-C-D+\pi}{2} - R_1\right) - Л_2\left(\tfrac{A-E-F+\pi}{2} + R_1\right) - \\ & - Л_2(P_2) + Л_2(Q_2) - Л_2(R_2) + Л_2(S_2) + \\ & + Л_2\left(\tfrac{B-C-A+\pi}{2} - Q_2\right) - Л_2\left(\tfrac{D-B-F+\pi}{2} + Q_2\right) - \\ & - Л_2\left(\tfrac{E-C-D+\pi}{2} - R_2\right) + Л_2\left(\tfrac{A-E-F+\pi}{2} + R_2\right) \end{aligned}$$

Volumes of hyperbolic tetrahedra

Formula of Milnor Let $S_\infty(z)$ be an ideal tetrahedra with vertices $\infty, 0, 1$ and $z \in \mathbb{C} \setminus \mathbb{R}$.



By decomposition into (three pairs of isometric) orthoschemes, Lobachevsky's formula yields

$$\mathrm{vol}_3(S_\infty(z)) = D(z) = \text{Л}_2(\alpha) + \text{Л}_2(\beta) + \text{Л}_2(\gamma)$$

$$\alpha = \arg z \quad , \quad \beta = \arg(1 - 1/z) \quad , \quad \gamma = \pi - (\alpha + \beta)$$

Hyperbolic volume in dimension three

Formula of Lobachevsky For an orthoscheme $R \subset H^3$,

$$\text{vol}_3(R) = \frac{1}{4}\Big\{ \text{Л}_2(\alpha+\theta) - \text{Л}_2(\alpha-\theta) + \text{Л}_2(\frac{\pi}{2}+\beta-\theta) + \text{Л}_2(\frac{\pi}{2}-\theta) +$$

$$+\text{Л}_2(\frac{\pi}{2}-\beta-\theta) + \text{Л}_2(\gamma+\theta) - \text{Л}_2(\gamma-\theta)\Big\} \quad ,$$

$$0 \le \theta = \arctan \frac{\left(\cos^2\beta - \sin^2\alpha \, \sin^2\gamma\right)^{1/2}}{\cos\alpha\cos\gamma} \le \frac{\pi}{2}$$

$$\text{Л}_2(x) = -\int_0^x \log|2\sin t|\, dt = -\int_0^x \log|1-\exp(2it)|\, dt$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sin(2kx)}{k^2} = \frac{1}{2}\,\text{Im}\,\text{Li}_2(e^{2ix})$$

denotes the (slightly modified) Lobachevsky function belonging to the family of **polylogarithms**

Polylogarithms

Classical polylogarithms (Leibniz, Johann Bernoulli)

$$\mathrm{Li}_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n} \quad , \quad z \in \mathbb{C} \quad \text{with} \quad |z| < 1$$

$$\mathrm{Li}_1(z) = -\log(1-z) \ ; \ \mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t)\, d\log t$$

Lobachevsky functions Let $\alpha \in \mathbb{R}$.

$$\mathrm{Jl}_2(\alpha) = \frac{1}{2} \mathrm{Im}\, \mathrm{Li}_2(e^{2i\alpha}) = \frac{1}{2} \sum_{r=1}^{\infty} \frac{\sin(2r\alpha)}{r^2}$$

$$\mathrm{Jl}_3(\alpha) = \frac{1}{4} \mathrm{Re}\, \mathrm{Li}_3(e^{2i\alpha}) = \frac{1}{4} \sum_{r=1}^{\infty} \frac{\cos(2r\alpha)}{r^3} \qquad \ldots$$

Modified polylogarithms (Bloch–Wigner, Goncharov...)

$$\mathcal{L}_1(z) = \mathrm{Re}\, \log z = \log |z|$$

$$D(z) = \mathcal{L}_2(z) = \mathrm{Im}\, \{ \mathrm{Li}_2(z) - \mathrm{Li}_1(z) \log |z| \}$$

$$\mathcal{L}_3(z) = \mathrm{Re}\, \{ \mathrm{Li}_3(z) - \mathrm{Li}_2(z) \log |z| + \tfrac{1}{3} \mathrm{Li}_1(z) \log^2 |z| \} \qquad \ldots$$

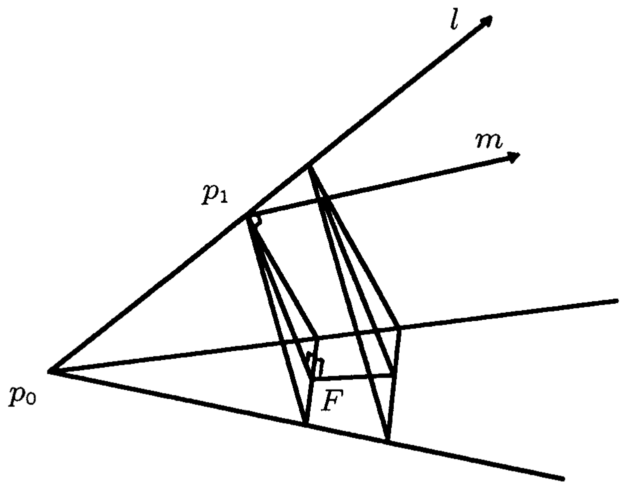
$$\sum_{k=1}^{5} (-1)^k \mathcal{L}_2(r_2(z_1, \ldots, \widehat{z_k}, \ldots, z_5)) = 0 \quad \text{for} \quad z_i \neq z_j$$

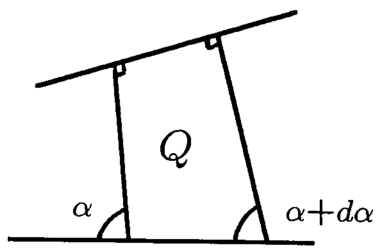
5–term relation of Spence–Abel

Proving Schläfli's Formula

- Let $S \in H^n$ be a simplex bounded by hyperplanes $H_0, \ldots, H_n$ with vertices p_i opposite to H_i for $0 \le i \le n$.

- Deform S by means of an infinitesimal parallel displacement of H_0 along the line $l = \cap_{i=2}^{n} H_i$. Then, only the dihedral angle $\alpha = \angle(H_0, H_1)$ attached at the codimension 2 face $F = S \cap H_0 \cap H_1$ varies.





$\mathrm{vol}_2(Q) = -d\alpha$

- Let m be the line through p_1 with $m \perp H_0$. An infinitesimal displacement along l moves H_0 into the same position as an infinitesimal displacement along m.

- $d\mathrm{vol}_n(S)$ =volume of the infinitesimal wedge determined by p_1, F and slice Q

$$\Longrightarrow \quad d\mathrm{vol}_n(S) = \tfrac{1}{n-1}\,\mathrm{vol}_{n-2}(F) \cdot \mathrm{vol}_2(Q)$$

$$= \tfrac{-1}{n-1}\,\mathrm{vol}_{n-2}(F)\; d\alpha$$

Schläfli's Differential Formula

THEOREM (Ludwig Schläfli, 1852)

Let $S \subset X^n_K$ denote a non-euclidean n-simplex with dihedral angles α_F at $(n-2)$-dimensional faces $F \subset S$. Then,

$$d\,\mathrm{vol}_n(S) = \frac{K}{n-1} \sum_F \mathrm{vol}_{n-2}(F)\, d\,\alpha_F \quad ,$$

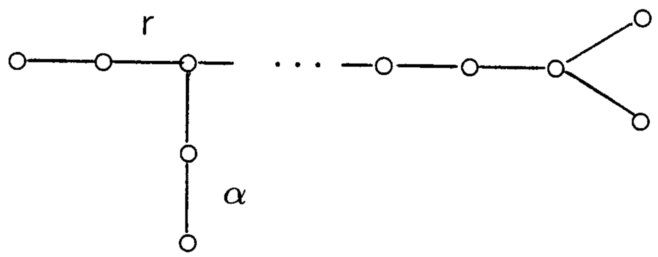
where $K \in \{\pm 1\}$ is the curvature of X^n_K and where $\mathrm{vol}_0(S) := 1$.

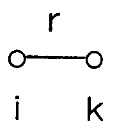
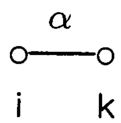
Consequence

Parity difference $n \equiv 1\,(2) \quad \longleftrightarrow \quad n \equiv 0(2)$

Coxeter–Vinberg–Schläfli schemes

A *scheme* Σ is a weighted graph, e.g. of the form



	hyperplane H
	$\angle(H_i, H_k) = \frac{\pi}{r}$ for $r > 2$
	$\angle(H_i, H_k) = \alpha$
	$\angle(H_i, H_k) = \frac{\pi}{3}$
	$H_i \perp H_k$

Important and simplest examples

Linear graphs of order $n+1$ are **orthoschemes** in X_K^n
for $K = +1$ iff Gram(Σ) is positive definit (Σ *elliptic*)
for $K = -1$ iff Gram(Σ) is of signature $(n, 1)$

Polytopes and orthoschemes

In $X^n_K \subset Y^{n+1}$ (vector space model)

$P = \cap_{i \in I} H_i^-$ **convex n–polytope**

H_i^- closed half-space bounded by $H_i = e_i^\perp$

Gram matrix $G(P) = (< e_i, e_j >_{Y^{n+1}})_{i,j \in I}$

A realisation criterion. (E. Vinberg)

Let $G = (g_{ij})$ be an indecomposable symmetric matrix of order m, of rank $n+1$ with $g_{ii} = 1$ and $g_{ij} \leq 0$ for $i \neq j$. Then G is the Gram $G(P)$ of an acute-angled polytope $P \subset X^n$ of finite volume defined uniquely up to isometry. In particular,

(1) if G is positive definite (elliptic), then $m = n+1$, and P is a simplex on the sphere S^n;

(2) if G is positive semidefinite (parabolic), then $m = n+2$, and P is a simplex in E^{n+1};

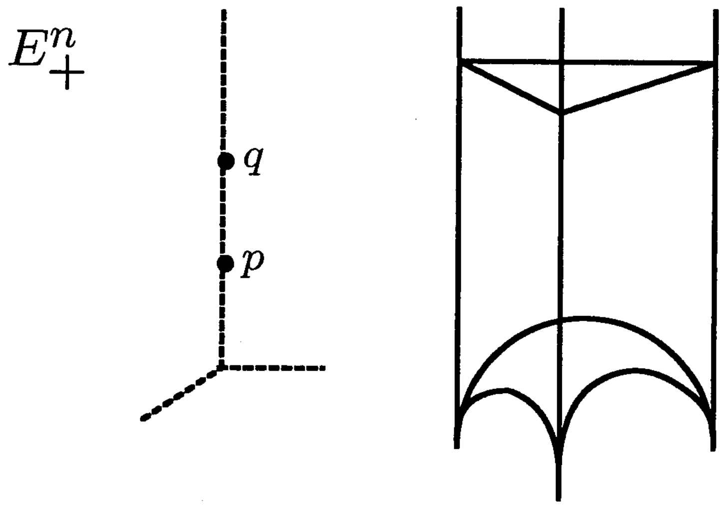
(3) if G is of signature $(n, 1)$ (hyperbolic), then P is a convex polytope in $\overline{H^n}$ with m facets.

If P has many right dihedral angles, i.e. for many i, j holds $\cos \alpha_{ij} = - < e_i, e_j > = 0$, then P best interpreted through its

Coxeter-Vinberg-Schläfli scheme $\Sigma(P)$

Models of hyperbolic geometry

- Lorentz–Minkowski vector space model $E^{n,1}$
- Poincaré upper half space model $H^n \subset E^n_+$



hyperbolic distance $d(p, q) = |\log \frac{p}{q}|$;