Quantum Invariant and Modular Form

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- quantum invariant for torus knot & link
- Eichler integral of modular form with half-integral weight

Introduction

- **Volume Conjecture** (Kashaev 1997)

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\left| \langle \mathcal{K} \rangle_N \right| = \mathrm{Vol}(S^3 \setminus \mathcal{K})$$

"Kashaev Invariant" $\langle \mathcal{K} \rangle_N$ for knot $\mathcal{K}$

$\Leftarrow$ quantum dilogarithm

- **Volume Conjecture** (Murakami–Murakami 2001)

$\langle \mathcal{K} \rangle_N$: N-colored Jones poly. at $q = \mathrm{e}^{2\pi \mathrm{i}/N}$

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \langle \mathcal{K} \rangle_N = v_3 \cdot \| S^3 \setminus \mathcal{K} \| + \mathrm{i}\, \mathrm{CS}(\mathcal{K})$$

- **Zagier's Identity** (Zagier 2001)

$$F(\mathrm{e}^{2\pi\mathrm{i}/N}) \simeq N^{3/2} \exp\left[-\frac{\pi\,\mathrm{i}}{12}\left(N - 3 + \frac{1}{N}\right)\right] + \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{2\,\pi}{\mathrm{i}\,N}\right)^k$$

$$b_0 = 1, \quad b_1 = 1, \quad b_2 = 3, \quad b_3 = 19, \quad \cdots$$

$$F(q) \equiv \sum_{n=0}^{\infty} (q)_n$$

(Kontsevich 1997)

❑ $F(1 - x) = \sum_{n=0}^{\infty} a_n\, x^n$

a_n: upper bound of linearly independent Vassiliev invariant of degree n

(Stoimenow 1998)

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 5, \quad \cdots$$

❑ *Nearly* Modular Property :

$$\widetilde{\Phi}(1/N) + (-\mathrm{i}\,N)^{3/2} \cdot \widetilde{\Phi}(-N) \simeq \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\pi}{12\,\mathrm{i}\,N}\right)^k$$

$$c_0 = 1,\ c_1 = 23,\ c_2 = 1681,\ c_3 = 257543, \cdots$$

$\widetilde{\Phi}(\tau) \equiv q^{1/24} F(q)$ is the Eichler integral of Dedekind η-function $\quad (q = \mathrm{e}^{2\pi\mathrm{i}\tau};\ \tau \in \mathbb{H})$

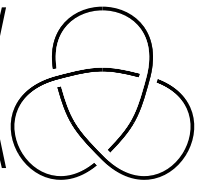
$$\eta(\tau) = q^{1/24}\,(q)_\infty$$

modular form with weight $1/2$

$$\eta(\tau + 1) = \mathrm{e}^{\pi\mathrm{i}/12}\,\eta(\tau)$$

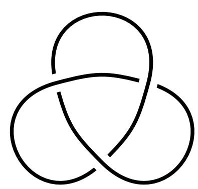
$$\eta(-1/\tau) = \sqrt{-\mathrm{i}\,\tau} \cdot \eta(\tau)$$

- Kashaev invariant ⇔ Zagier's identity

$\Big\langle$  $\Big\rangle_N = \sum_{n=0}^{N} (\omega)_n = F(\omega)$ $\quad \omega = \mathrm{e}^{2\pi \mathrm{i}/N}$

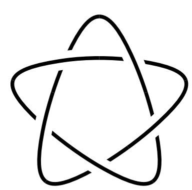
- <u>This Talk</u>

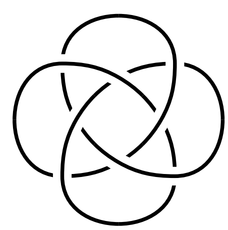
Trefoil



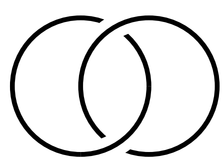
⇓

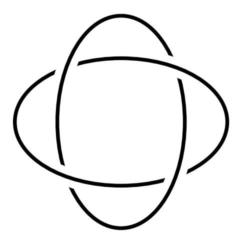
(s,t) Torus Knot





$(2,2m)$ Torus Link





Eichler integral

$$F(\tau) = \sum_{n=1}^{\infty} a_n \, q^n$$

modular, weight $k \in \mathbb{Z}_{\geq 2}$

Eichler integral is defined as $k-1$ integrations of $F(\tau)$ w.r.t. τ

$$\widetilde{F}(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} \, q^n$$

We have

$$(c\,\tau + d)^{k-2} \cdot \widetilde{F}(\gamma(\tau)) - \widetilde{F}(\tau) = G_\gamma(\tau)$$

where $\gamma \in SL(2;\mathbb{Z})$

$$G_\gamma(z) = \frac{(2\,\pi\,\mathrm{i})^{k-1}}{(k-2)!} \int_{\gamma^{-1}(\infty)}^{\infty} F(\tau)\,(z-\tau)^{k-2}\,\mathrm{d}\tau$$

$\Rightarrow$ Generalization to half-integral weight

(Lawrence–Zagier 1999, Zagier 2001)

(s,t)-Torus Knot

- $\mathcal{K}$: (s,t)-Torus Knot

  (s, t : coprime integers)

  $J_N(\mathcal{K})$: N-colored Jones polynomial

$$2\,\mathrm{sh}\left(\frac{N\hbar}{2}\right)\frac{J_N(\mathcal{K})}{J_N(\mathcal{O})} = \mathrm{e}^{-\frac{\hbar}{4}\left(\frac{t}{s}+\frac{s}{t}\right)}$$

$$\times \sum_{\varepsilon=\pm 1}\sum_{k=-(N-1)/2}^{(N-1)/2} \varepsilon\,\exp\left[\hbar\,s\,t\left(k+\frac{s+\varepsilon\,t}{2\,s\,t}\right)^2\right]$$

(Morton 1995, Rosso–Jones 1993)

$\Longrightarrow \langle\mathcal{K}\rangle_N$ (Kashaev–Tirkkonen 1999)

- Define for $1 \le n \le s-1, \quad 1 \le m \le t-1$

$$\boxed{\Phi_{s,t}^{(n,m)}(\tau) = \sum_{k=0}^{\infty}\chi_{2st}^{(n,m)}(k)\,q^{k^2/4st}}$$

$$\chi_{2st}^{(n,m)}(k) = \begin{cases} +1 & k = \pm(n\,t - m\,s) \quad \mathrm{mod}\ 2\,s\,t \\ -1 & k = \pm(n\,t + m\,s) \quad \mathrm{mod}\ 2\,s\,t \\ 0 & \text{otherwise} \end{cases}$$

✎ **Remark:** CFT Minimal Model $\mathcal{M}(s,t)$

(BPZ 1984)

$$c(s,t) = 1 - \frac{6\,(s-t)^2}{s\,t}$$

$$\Delta^{s,t}_{n,m} = \frac{(n\,t - m\,s)^2 - (s-t)^2}{4\,s\,t}$$

Virasoro Character (Rocha-Caridi 1985)

$$\boxed{\mathrm{ch}^{s,t}_{n,m}(\tau) = \mathrm{Tr}\, q^{L_0 - c(s,t)/24} = \frac{\Phi^{(n,m)}_{s,t}(\tau)}{\eta(\tau)}}$$

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c(s,t)}{12}\,n\,(n^2-1)\,\delta_{n+m,0}$$

highest weight state : $|\Delta^{s,t}_{n,m}\rangle$

$$\mathrm{ch}^{s,t}_{n,m}(\tau) = \mathrm{ch}^{s,t}_{s-n,t-m}(\tau)$$

$$\Rightarrow D(s,t) = \frac{1}{2}\,(s-1)\,(t-1)$$

- Modular covariant: weight $1/2$

(Cappelli–Itzykson–Zuber 1987, ...)

$$\Phi_{s,t}^{(n,m)}(\tau+1) = \mathrm{e}^{\frac{(nt-ms)^2}{2st}\pi \mathrm{i}}\, \Phi_{s,t}^{(n,m)}(\tau)$$

$$\Phi_{s,t}^{(n,m)}(\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \sum_{n',m'} \mathbf{S}_{n,m}^{n',m'}\, \Phi_{s,t}^{(n',m')}(-1/\tau)$$

$$\mathbf{S}_{n,m}^{n',m'} = \sqrt{\frac{8}{s\,t}}\,(-1)^{nm'+mn'+1} \times \sin\left(n\,n'\,\frac{t}{s}\,\pi\right)\,\sin\left(m\,m'\,\frac{s}{t}\,\pi\right)$$

- Eichler integral

$$\widetilde{\Phi}_{s,t}^{(n,m)}(\tau) = -\frac{1}{2}\sum_{k=0}^{\infty} k\,\chi_{2st}^{(n,m)}(k)\, q^{k^2/4st}$$

We take a limit $\tau \to 1/N$

In terms of L-function, we have in $y \searrow 0$

$$\widetilde{\Phi}_{s,t}^{(n,m)}\left(\frac{1}{N} + \mathrm{i}\,\frac{y}{2\,\pi}\right)$$

$$\simeq -\frac{1}{2}\sum_{k=0}^{\infty}\frac{L_\omega(-2\,k-1,\chi_{2st}^{(n,m)})}{k!}\left(-\frac{y}{4\,s\,t}\right)^k$$

$$L_\omega(k,\chi_{2st}^{(n,m)}) = \frac{1}{(2stN)^k}\sum_{j=1}^{2stN}\chi_{2st}^{(n,m)}(j)\,\mathrm{e}^{\frac{j^2}{2stN}\pi\mathrm{i}}\,\zeta\left(k,\frac{j}{2stN}\right)$$

which, by analytic continutation, gives

$$\boxed{\widetilde{\Phi}_{s,t}^{(n,m)}(1/N) = \frac{s\,t\,N}{2}\sum_{k=1}^{2stN}\chi_{2st}^{(n,m)}(k)\,\mathrm{e}^{\frac{k^2}{2stN}\pi\mathrm{i}}\,B_2\left(\frac{k}{2\,s\,t\,N}\right)}$$

Bernoulli poly: $\displaystyle \frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t-1} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\,B_k(x)$

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6}$$

$$\boxed{\langle\mathcal{K}\rangle_N = \mathrm{e}^{-\frac{(st-s-t)^2}{2stN}\pi\mathrm{i}}\cdot\widetilde{\Phi}_{s,t}^{(s-1,1)}(1/N)}$$

- Nearly modular property

$$\hat{\Phi}_{s,t}^{(n,m)}(z) + \left(\frac{1}{\mathrm{i}\,z}\right)^{3/2} \sum_{n',m'} \mathbf{S}_{n,m}^{n',m'}\, \hat{\Phi}_{s,t}^{(n',m')}(-1/z) = r_{s,t}^{(n,m)}(z;0)$$

$$\hat{\Phi}_{s,t}^{(n,m)}(z) = \sqrt{\frac{s\,t\,\mathrm{i}}{8\,\pi^2}} \int_{z^*}^{\infty} \frac{\Phi_{s,t}^{(n,m)}(\tau)}{(\tau - z)^{3/2}}\,\mathrm{d}\tau$$

$$r_{s,t}^{(n,m)}(z;\alpha) = \sqrt{\frac{s\,t\,\mathrm{i}}{8\,\pi^2}} \int_{\alpha}^{\infty} \frac{\Phi_{s,t}^{(n,m)}(\tau)}{(\tau - z)^{3/2}}\,\mathrm{d}\tau$$

Asymptotics in $N \to \infty$: $\widetilde{\Phi}_{s,t}^{(n,m)}(1/N) = \hat{\Phi}_{s,t}^{(n,m)}(1/N)$

$$\widetilde{\Phi}_{s,t}^{(n,m)}(1/N) + (-\mathrm{i}\,N)^{3/2} \sum_{n',m'} \mathbf{S}_{n,m}^{n',m'}\, \phi_{s,t}(n',m')\, \mathrm{e}^{-\frac{(n' t - m' s)^2}{2st}\pi \mathrm{i} N} \simeq \sum_{k=0}^{\infty} \frac{T_{s,t}^{(n,m)}(k)}{k!} \left(\frac{\pi}{2\,s\,t\,\mathrm{i}\,N}\right)^k$$

$$\phi_{s,t}(n,m) = \begin{cases} (s-n)\,m, & \text{if } n\,t > m\,s, \\ n\,(t-m), & \text{if } n\,t < m\,s, \end{cases}$$

$$T_{s,t}^{(n,m)}(k) = \frac{1}{2}\,(-1)^{k+1}\, L(-2\,k-1, \chi_{2st}^{(n,m)})$$

- **Alexander Poly** for $\mathcal{K} = (s, t)$ Torus Knot

$$A_{\mathcal{K}}(z) = \frac{(z^{1/2} - z^{-1/2})\,(z^{s\,t/2} - z^{-s\,t/2})}{(z^{s/2} - z^{-s/2})\,(z^{t/2} - z^{-t/2})}$$

Tail of asymptotic expansion of $\langle\mathcal{K}\rangle_N$

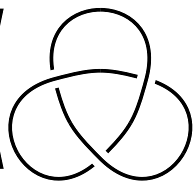
$$\frac{\sin(s\,x)\sin(t\,x)}{\sin(s\,t\,x)} = \sum_{k=0}^{\infty} \frac{T_{s,t}^{(s-1,1)}(k)}{(2\,k+1)!}\,x^{2k+1}$$

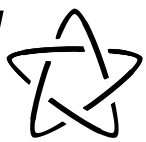
$$\Updownarrow$$

$$\frac{z^{1/2} - z^{-1/2}}{A_{\mathcal{K}}(z)}$$

- q-series

  We compute $\langle \mathcal{K} \rangle_N$ using Kashaev R-matrix

✦ $\Big\langle$  $\Big\rangle_N = \sum_{n=0}^{N} (\omega)_n$

✦ $\Big\langle$  $\Big\rangle_N = \sum_{n=0}^{N} (\omega)_n \sum_{c=0}^{n} \omega^{c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}_\omega$

✦ $\cdots\cdots$

We regard these expresssions as reduction of infinite (formal) q-series $q \to \omega \equiv \mathrm{e}^{2\pi \mathrm{i}/N}$

★ q-series identities $\qquad [q = \mathrm{e}^{-t} \text{ in } t \searrow 0]$

$$\sum_{n=0}^{\infty} (q)_n = \mathrm{e}^{t/24} \sum_{k=0}^{\infty} \frac{T_{2,3}^{(1,1)}(k)}{k!} \left(\frac{t}{24}\right)^k$$

$$\sum_{n=0}^{\infty} (q)_n \sum_{c=0}^{n} q^{c(c+1)} \begin{bmatrix} n \\ c \end{bmatrix}_q = \mathrm{e}^{9\,t/40} \sum_{k=0}^{\infty} \frac{T_{2,5}^{(1,1)}(k)}{k!} \left(\frac{t}{40}\right)^k$$

$$\sum_{n=0}^{\infty} (q)_n \sum_{c=0}^{n+1} q^{c^2} \begin{bmatrix} n+1 \\ c \end{bmatrix}_q = \mathrm{e}^{t/40} \sum_{k=0}^{\infty} \frac{T_{2,5}^{(1,2)}(k)}{k!} \left(\frac{t}{40}\right)^k$$

★ Rogers–Ramanujan identity

$$(q)_\infty \sum_{c=0}^{\infty} \frac{q^{c(c+1)}}{(q)_c} = (q, q^4, q^5; q^5)_\infty \propto \Phi_{2,5}^{(1,1)}(\tau)$$

$$(q)_\infty \sum_{c=0}^{\infty} \frac{q^{c^2}}{(q)_c} = (q^2, q^3, q^5; q^5)_\infty \propto \Phi_{2,5}^{(1,2)}(\tau)$$

$(2, 2m+1)$-Torus Knot

$\Leftrightarrow$ Gordon–Andrews Identity

$\Leftrightarrow$ Eichler of $(q, q^{2m}, q^{2m+1}; q^{2m+1})_\infty$

$(2, 2\,m)$-Torus Link

- $\mathcal{K}$: $(2, 2m)$-Torus Link

  $J_N(\mathcal{K})$: N-colored Jones polynomial

$$2\,\mathrm{sh}\left(\frac{N\,\hbar}{2}\right)\frac{J_N(\mathcal{K})}{J_N(\mathcal{O})} = \mathrm{e}^{-\hbar\,(N^2-1)\,m/2}$$

$$\times \sum_{\varepsilon=\pm 1}\sum_{k=0}^{N-1}\varepsilon\,\exp\left[\hbar\left(m\,k^2 + (m+\varepsilon)\,k + \frac{1}{2}\,\varepsilon\right)\right]$$

  which gives $\langle\mathcal{K}\rangle_N$

- Define for $0 \leq a \leq m-2$

$$\boxed{\Phi_m^{(a)}(\tau) = \sum_{k\in\mathbb{Z}} k\,\chi_{2m}^{(a)}(k)\;q^{k^2/4m}}$$

$$\chi_{2m}^{(a)}(k) = \begin{cases} +1 & \text{for } k = m-(1+a) \mod 2\,m \\ -1 & \text{for } k = m+(1+a) \mod 2\,m \\ 0 & \text{otherwise} \end{cases}$$

✎ **Remark**: $\hat{su}(2)_{m-2}$ character (Kac)

$$\mathrm{ch}_{\lambda}^{m-2}(\tau) = \frac{\Phi_m^{(m-2-\lambda)}(\tau)}{2\left[\eta(\tau)\right]^3}$$

- Modular Form with weight $3/2$

(Kac–Peterson 1984)

$$\Phi_m^{(m-1-a)}(\tau+1) = \mathrm{e}^{\frac{a^2}{2m}\pi\mathrm{i}}\,\Phi_m^{(m-1-a)}(\tau)$$

$$\Phi_m^{(m-1-a)}(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{3/2} \sum_{b=1}^{m-1} \mathbf{M}_{a,b}\,\Phi_m^{(m-1-b)}(-1/\tau)$$

$$\mathbf{M}_{a,b} = \sqrt{\frac{2}{m}}\,\sin\left(\frac{a\,b}{m}\,\pi\right)$$

- Eichler integral

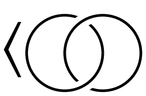
$$\widetilde{\Phi}_m^{(a)}(\tau) = m \sum_{k=0}^{\infty} \chi_{2m}^{(a)}(k)\, q^{k^2/4m}$$

Using analytic continuation for L-function, we have in a limit $\tau \to {}^1\!/_N$

$$\boxed{\widetilde{\Phi}_m^{(a)}\left({}^1\!/_N\right) = -m \sum_{k=1}^{2mN} \chi_{2m}^{(a)}(k)\, \mathrm{e}^{\frac{k^2}{2mN}\pi \mathrm{i}}\, B_1\left(\frac{k}{2\,m\,N}\right)}$$

- After some calculations, we find that

$$\boxed{\langle \mathcal{K} \rangle_N = N\, \mathrm{e}^{-\frac{(m-1)^2}{2mN}\pi \mathrm{i}} \cdot \widetilde{\Phi}_m^{(0)}\left({}^1\!/_N\right)}$$

$\langle$  $\rangle_N = N$

- Nearly modular property

$$\hat{\Phi}_m^{(a)}(z) = \sqrt{\frac{1}{\mathrm{i}\,z}} \sum_{b=1}^{m-1} \mathbf{M}_{m-1-a,b}\, \hat{\Phi}_m^{(m-1-b)}(-1/z) + r_m^{(a)}(z;0)$$

$$\hat{\Phi}_m^{(a)}(z) = \sqrt{\frac{m}{8\,\mathrm{i}}} \int_{z^*}^{\infty} \frac{\Phi_m^{(a)}(\tau)}{\sqrt{\tau - z}}\, \mathrm{d}\tau$$

$$r_m^{(a)}(z;\alpha) = \sqrt{\frac{m}{8\,\mathrm{i}}} \int_{\alpha}^{\infty} \frac{\Phi_m^{(a)}(\tau)}{\sqrt{\tau - z}}\, \mathrm{d}\tau$$

We get asymptotic expansion in $N \to \infty$

$$\widetilde{\Phi}_m^{(a)}(1/N) \simeq \sqrt{-\mathrm{i}\,N} \sum_{b=1}^{m-1} \mathbf{M}_{m-1-a,b}\,(m-b)\,\mathrm{e}^{-\frac{b^2}{2m}\pi \mathrm{i} N} + \sum_{k=0}^{\infty} \frac{E_m^{(a)}(k)}{k!} \left(\frac{\pi}{2\,m\,\mathrm{i}\,N}\right)^k$$

$$E_m^{(a)}(k) = m\,(-1)^k\, L(-2\,k, \chi_{2m}^{(a)})$$

- **Alexander Poly** for $\mathcal{K} = (2, 2m)$ Torus Link

$$A_{\mathcal{K}}(z) = \frac{z^{m} - z^{-m}}{z^{1/2} + z^{-1/2}}$$

Tail of asymptotic expansion of $\langle \mathcal{K} \rangle_N$

$$\frac{m \ \sin(x)}{\sin(m\,x)} = \sum_{k=0}^{\infty} \frac{E_m^{(0)}(k)}{(2\,k)!}\, x^{2k}$$

$$\Updownarrow$$

$$\frac{z^{1/2} - z^{-1/2}}{A_{\mathcal{K}}(z)}$$

Conclusion

☞ $\langle\mathcal{K}\rangle_N$ coincides with a limit $\tau \to 1/N$ of the *Eichler integral* of modular form with weight $\begin{Bmatrix} 1/2 \\ 3/2 \end{Bmatrix}$ when $\mathcal{K}$ is $\begin{Bmatrix} (s,t)\text{-torus knot} \\ (2,2m)\text{-torus link} \end{Bmatrix}$

☞ Exact *asymptotic expansion* in $N \to \infty$ of Kashaev's invarint $\langle\mathcal{K}\rangle_N$ (N-colored Jones polynomial at $q = \mathrm{e}^{2\pi \mathrm{i}/N}$) for (s,t)-torus knot & $(2,2m)$-torus link.

☞ Tail of asymptotic expansion of $\langle\mathcal{K}\rangle_N$ is related to an inverse of the *Alexander polynomial*

☞ References:

✔ `http://gogh.phys.s.u-tokyo.ac.jp/~hikami`

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