On the volume conjecture for quantum $6j$ symbols

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Jones Polynomial

- Jones polynomial (1984) $\longleftarrow \mathcal{U}_q(sl_2)$

$$t^{-1}\, V_{K_-}(t) - t\, V_{K_+}(t) = \left(t^{1/2} - t^{-1/2}\right)\, V_{K_0}$$

Cf. Alexander polynomial

$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\, \nabla_{K_0}(z)$$

- HOMFLY-PT polynomial (1987) $\longleftarrow \mathcal{U}_q(sl_n)$

$$a^{-1}\, P_{K_-}(t) - a\, P_{K_+}(t) = \left(t^{1/2} - t^{-1/2}\right)\, P_{K_0}$$

- Kauffman polynomial (1987) $\longleftarrow \mathcal{U}_q(so_n),\ \mathcal{U}_q(sp_{2n})$

$$a^{-1}\, F_{|K_-|}(t) - a\, F_{|K_+|}(t) = \left(t^{1/2} - t^{-1/2}\right)\left(F_{|K_0|} - F_{|K_\infty|}\right)$$

Overview

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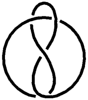
Kashaev's conjecture

Let L be a knot and $K_N(L)$ be the quantum invariant introduced by Kashaev. Then

$$\lim_{N\to\infty}\frac{2\pi}{N}\log|K_N(L)| = \mathrm{Vol}(S^3\setminus L).$$

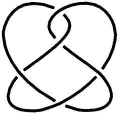
4_1 :



$$K_N(4_1) = \sum_{k=0}^{N-1}|(q)_k|^2, \qquad (q)_k = \prod_{j=1}^{k}(1-q^j)$$

$$\xrightarrow[N\to\infty]{} \mathbf{2.02988321\ldots}$$

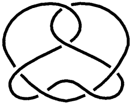
5_2 :



$$K_N(5_2) = \sum_{k\le l}\frac{q^{-k(l+1)/2}\,(q)_l^2}{(q^{-1})_k}$$

$$\xrightarrow[N\to\infty]{} \mathbf{2.82812208\ldots}$$

6_1 :



$$K_N(6_1) = \sum_{k+l\le m}\frac{q^{(m-k-l)(m-k+1)/2}\,|(q)_m|^2}{(q)_k(q^{-1})_l}$$

$$\xrightarrow[N\to\infty]{} \mathbf{3.16396322\ldots}$$

$K_N(L)$ is equal to the colored Jones inv. $V_L^N(q)$ for N-dim. rep. of $\mathcal{U}_q(sl_2)$ at $q=\exp 2\pi\sqrt{-1}/N$. (**H. Murakami-J.M.**)

Generalizations of Kashaev's Conjecture

$s_N = \exp(\pi\sqrt{-1}/N), \qquad q_N = s_N^2.$

Volume Conjecture. (**H. Murakami-J.M.**)

L : a knot

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left| V_L^N(q_N) \right| = \mathrm{Vol}_{Gr}(S^3 \setminus L),$$

where Vol_{Gr} is Gromov's simplicial volume.

Complesified Volume Conjecture
(**H.Murakami-J.M.-M.Okamoto-T.Takata-Y.Yokota**)

L : hyperbolic knot

$$\lim_{N\to\infty} \frac{2\pi}{N} \log V_L^N(q_N) = \mathrm{Vol}(S^3 \setminus L) + \sqrt{-1}\,\mathrm{CS}(S^3 \setminus L).$$

Proof for the figure-eight knot (**Ekholm**)

K : figure-eight knot, $s_N = \exp(\pi\sqrt{-1}/N)$, $q_N = s_N^2$.

$$V_K^N(s_N) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} (s_N^{N-k} - s_N^{-N+k})(s_N^{N+k} - s_N^{-N-k}) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} 4\sin^2 \frac{\pi k}{N}.$$

Let $a_j = \prod_{k=1}^{j} 4\sin^2 \frac{\pi k}{N}$ and $a_{j_{\max}}$ be the max. term of a_j.
Since $a_{j_{\max}} \le V_K^N(s_N) \le N\, a_{j_{\max}}$,

$$\lim_{N\to\infty} \frac{2\pi \log a_{j_{\max}}}{N} \le \lim_{N\to\infty} \frac{2\pi \log V_K^N(s_N)}{N} \le \lim_{N\to\infty} \frac{2\pi \log(N\, a_{j_{\max}})}{N} = \lim_{N\to\infty} \frac{2\pi \log(a_{j_{\max}})}{N}.$$

a_j is decreasing for small j's, increasing for middle j's and then decreasing for large j's. It takes the maximal at $j \fallingdotseq \frac{5N}{6}$. Since $a_0 = 1$, $a_{N-1} = N^2$, a_j is maximum at $j \fallingdotseq \frac{5N}{6}$.
Therefore

$$\lim_{N\to\infty} \frac{2\pi \log(a_{j_{\max}})}{N} = \lim_{N\to\infty} \frac{4\pi \sum_{k=1}^{5N/6} \log(2\sin \pi k/N)}{N} =$$

$$4\int_0^{\frac{5\pi}{6}} \log(2\, sinx)\, dx = -4\Lambda\left(\frac{5\pi}{6}\right) = \mathbf{2.02988321\ldots}$$

Volume potential function

- **Dilogarithm function**

  Analytic continuation of

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-u)}{u}\,du = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots \quad (0 < x < 1).$$

  It is a multi-valued function and its branches are

$$\mathrm{li}_2(z) = \mathrm{Li}_2(z) + 2\,k\,\pi\,\sqrt{-1}\,\log z + 4\,l\,\pi^2 \qquad (k, l \in \mathbb{Z})$$

- **Volume potential function** $U(x_1, x_2, \dots)$

  The function obtained by replacing $(q)_k$ in $\mathcal{U}_q(sl_2)$-invariant by $\mathrm{Li}_2(x)$ $(x = q_N^k)$.

- **Saddle points of the volume potential function**

  Points satisfying $\dfrac{\partial U}{\partial x_i} = 0$ $(i = 1, 2, \dots)$.

  These equations correspond to the glueing equation of the tetrahedral decomposition (Yokota).

Hyperbolic volume from the saddle point

K : hyperbolic knot,
U : Volume potential function of $V_K^N(q_N)$ (colored Jones)

- **Hyperbolic volume**

  For some $x_1^{(0)}$, $x_2^{(0)}$, ... satisfying $\exp\left(\frac{\partial U}{\partial x_i}\right) = 1$,

$$U(x_1^{(0)}, x_2^{(0)}, \dots) - \sum_i \frac{\partial U}{\partial x_i}\bigg|_{x_1 = x_1^{(0)}, \cdots} \log x_i^{(0)}$$

  is the hyperbolic volume of the complement of K (**Yokota, J. Cho**).

- **Optimistic calculation** (**H.Murakami**, arXiv:math/0005289)

  M : 3-manifold, $\tau_N(M)$: Witten-Reshetikhin-Turaev invariant

  Check that $\tau_N(M) \longrightarrow$ hyperoblic volume of M for M obtained by surgery along the figure-eight knot.

- **Optimistic conjecture**

  $\mathcal{U}_q(sl_2)$ inariant $\longrightarrow$ hyperbolic volume

$q_N \longrightarrow q_N^2$

Attention!

- The $\mathcal{U}_q(sl_2)$-invariant grows exponentially **only** for Kashaev's invariant and its deformation.
- For other $\mathcal{U}_q(sl_2)$-invariants, they does not grow exponentially.

Renovation by **Q. Chen–T. Yang** arXiv:1503.02547

- Replace $q_N = \exp(2\pi\sqrt{-1}/N)$ by q_N^2.
- Various $\mathcal{U}_q(sl_2)$-invariants grow exponentially and the growth rates are given by the hyperbolic volume of the corresponding geometric objects.

  Including:

  WRT invariant, Turaev-Viro inv. for 3-mfds,
  Kirillov-Reshetikhin invariant for knotted graphs

Overview

Quantum Invariants

Volume Conjecture

Quantum $6j$ symbol

Quantum 6j symbol

The quantum 6j symbol is introduced to express

$$\begin{pmatrix} V_m \hookrightarrow & & \\ & V_l \otimes V_k \hookrightarrow & \\ & & V_i \otimes V_j \otimes V_k \end{pmatrix} = \sum_n \begin{Bmatrix} i & j & l \\ m & k & n \end{Bmatrix}_q \begin{pmatrix} V_m \hookrightarrow & & \\ & V_i \otimes V_n \hookrightarrow & \\ & & V_i \otimes V_j \otimes V_k \end{pmatrix}$$

for representations of $\mathcal{U}_q(sl_2)$ by **Kirillov-Reshetikhin**.

$$\{k\} = \{q^{1/2} - q^{-1/2}\}, \qquad \{k\}! = \{k\}\,\{k-1\}\ \dots\ \{1\},$$

$$W_1(e) = \frac{\{c+1\}}{\{1\}}, \qquad W_2(f) = \frac{\{\frac{c_1+c_2+c_3}{2}+1\}!}{\{1\}\,\{\frac{-c_1+c_2+c_3}{2}\}!\{\frac{c_1-c_2+c_3}{2}\}!\{\frac{c_1+c_2-c_3}{2}\}!},$$

$$\begin{Bmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{Bmatrix}_q^{RW} = \frac{\begin{Bmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{Bmatrix}_q}{\sqrt{W_1(c_5)\,W_1(c_6)}} = \frac{\begin{pmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{pmatrix}}{\sqrt{W_2(1{,}2{,}5)\,W_2(1{,}3{,}6)\,W_2(2{,}4{,}6)\,W_2(3{,}4{,}5)}},$$

$$\begin{pmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{pmatrix} = \sum_{k=\max(B_1,\cdots,B_4)}^{\min(A_1,\dots,A_3)} \frac{(-1)^k\,\{k+1\}!}{\{1\}\,\{A_1-k\}!\ \dots\ \{A_3-k\}!\,\{k-B_1\}!\ \dots\ \{k-B_4\}!},$$

where

$$\begin{aligned} 2A_1 &= c_1+c_2+c_3+c_4, & 2B_1 &= c_1+c_2+c_5, \\ 2A_2 &= c_1+c_4+c_5+c_6, & 2B_2 &= c_1+c_3+c_6, \\ 2A_3 &= c_2+c_3+c_5+c_6, & 2B_3 &= c_2+c_4+c_6, \\ & & 2B_4 &= c_3+c_4+c_5. \end{aligned}$$

Here we assume $c_1, \cdots, c_6$ are admissible, i.e. $A_i,\ B_j,\ A_i - B_j \in \mathbb{Z}_{\geq 0}$.

Turaev-Viro invariant

State sum on a tetrahedral decomposition

Let $s_N = \exp(\frac{\pi\sqrt{-1}}{N})$, $[k] = \frac{s_N^k - s_N^{-k}}{s_N - s_N^{-1}}$, $[k]! = [k][k-1]\cdots[1]$, $I_N = \{0, 1, \ldots, N-2\}$.

- **Turaev-Viro invariant**

  Let M be a closed 3-manifold and let Δ be a tetrahedral decomposition of M. Let T, F, E be the set of tetrahedrons, faces, edges of Δ. Then

  $$\mathrm{TV}_N(M) = \sum_{\substack{\varphi\,:\,E \to I_N \\ \text{admissible}}} \prod_{e \in E} W_1(e) \prod_{f \in F} W_2(f)^{-1} \prod_{t \in T} W_3(t).$$

  TV $\to$ potential fun. $\to$ saddle pt. $\to$ hyp. volume

- **Quantum** $6j$ **symbol**

  quantum $6j$ symbol $\to$ potential function $\to$ saddle point $\to$ hyperbolic volume of tetrahedron

  (**J. M.-M. Yano**)

Chen-Yang's invariant and $q \longrightarrow q^2$

Let $s_N = \exp(\pi\sqrt{-1}/N)$ and $q_N = s_N^2$.

- **Chen-Yang's invariant**

  Extend TV-invariant for 3-manifolds with boundary by using ideal tetrahedrons and truncated tetrahedrons.

Conjecture (Q.Chen-T.Yang, arXiv:1503.02547)
Let M be a 3-manifold, N be a positive odd integer, $CY_N(M)$ be Chen-Yang's invariant, $\tau_N(M)$ be the WRT invariant, and $TV_N(M)$ be the Turaev-Viro invariant of M. Then we have

1. $\lim_{N\to\infty} \frac{2\pi}{N} \log\left(CY_N(M)|_{s_N \to s_N^2} \right) = \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)$
2. $\lim_{N\to\infty} \frac{4\pi}{N} \log\left(\tau_N(M)|_{s_N \to s_N^2} \right) = \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)$
3. $\lim_{N\to\infty} \frac{2\pi}{N} \log\left(TV_N(M)|_{s_N \to s_N^2} \right) = \mathrm{Vol}(M)$,
   since 2. and the fact that $TV_N(M) = |\tau_N(M)|^2$.

Volume conjecture for the quantum 6j symbol

Conjecture

T : hyperbolic tetrahedron with dihedral angles $\theta_1, \ldots, \theta_6$,
N : positive odd integer,
$a_i^{(N)}$: admissible sequences with $\lim_{N\to\infty} \frac{2\pi}{N} a_i^{(N)} = \pi - \theta_i$,
$(1 \leq i \leq 6)$

Then

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left| \begin{Bmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{Bmatrix}^{RW}_{q_N^2} \right| = \mathrm{Vol}(T).$$

Theorem

The above conjecture is true if all the vertices are truncated, i.e. the sum of the three dihedral angles sharing a vertex is less than π.

Proof

Use the same idea for the proof for figure-eight knot. If the terms of the sum are all positive, then only the maximum term contributes to the limit.

Recall

$$\begin{Bmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{Bmatrix}_q^{RW} = \frac{\begin{pmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{pmatrix}}{\sqrt{W_2(c_1,c_2,c_5)\,W_2(c_1,c_3,c_6)\,W_2(c_2,c_4,c_6)\,W_2(c_3,c_4,c_5)}},$$

$$W_2(f) = \frac{\{\frac{c_1+c_2+c_3}{2}+1\}!}{\{1\}\,\{\frac{-c_1+c_2+c_3}{2}\}!\{\frac{c_1-c_2+c_3}{2}\}!\{\frac{c_1+c_2-c_3}{2}\}!},$$

$$\begin{pmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{pmatrix} = \sum_{k=\max(B_1,\cdots,B_4)}^{\min(A_1,\ldots,A_3)} \frac{(-1)^k\,\{k+1\}!}{\{1\}\,\{A_1-k\}!\ldots\{A_3-k\}!\,\{k-B_1\}!\ldots\{k-B_4\}!},$$

where

$$\begin{aligned} 2A_1 &= c_1+c_2+c_3+c_4, & 2B_1 &= c_1+c_2+c_5, \\ 2A_2 &= c_1+c_4+c_5+c_6, & 2B_2 &= c_1+c_3+c_6, \\ 2A_3 &= c_2+c_3+c_5+c_6, & 2B_3 &= c_2+c_4+c_6, \\ & & 2B_4 &= c_3+c_4+c_5. \end{aligned}$$

Now apply the above idea to $\begin{pmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{pmatrix}$.

Proof (sign)

$$\begin{pmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{pmatrix} = \sum_{k=\max(B_1,\cdots,B_4)}^{\min(A_1,\dots,A_3)} \alpha_k$$

where

$$\alpha_k = \frac{(-1)^k \{k+1\}!}{\{1\}\{A_1-k\}!\dots\{A_3-k\}!\{k-B_1\}!\dots\{k-B_4\}!},$$

$$2A_1 = a_1^{(N)} + a_2^{(N)} + a_3^{(N)} + a_4^{(N)}, \quad 2B_1 = a_1^{(N)} + a_2^{(N)} + a_5^{(N)},$$
$$2A_2 = a_1^{(N)} + a_4^{(N)} + a_5^{(N)} + a_6^{(N)}, \quad 2B_2 = a_1^{(N)} + a_3^{(N)} + a_6^{(N)},$$
$$2A_3 = a_2^{(N)} + a_3^{(N)} + a_5^{(N)} + a_6^{(N)}, \quad 2B_3 = a_2^{(N)} + a_4^{(N)} + a_6^{(N)},$$
$$2B_4 = a_3^{(N)} + a_4^{(N)} + a_5^{(N)}.$$

Recall that

$$s_N = \exp\frac{\pi\sqrt{-1}}{N}, \quad \{j\} = s_N^{2j} - s_N^{-2j} = 2\sqrt{-1}\sin\frac{2j\pi}{N}, \quad \begin{cases} \mathrm{Im}\{j\} > 0 & \text{if } 0 < j < \frac{N}{2}, \\ \mathrm{Im}\{j\} < 0 & \text{if } \frac{N}{2} < j < N. \end{cases}$$

Since $\lim_{N\to\infty} \frac{2\pi}{N} a_i^{(N)} = \pi - \theta_i$ and $\theta_1 + \theta_2 + \theta_5 \leq \pi, \cdots$, we have $B_i > N/2$, $A_i < N$ and the terms of the denominator of α_k are all positive. Hence the range of k is contained in $N/2 < k < N$ and

$$\operatorname{sign}\frac{\{k+1\}!}{\sqrt{-1}^{k+1}} = (-1)^{(N+1)/2}.$$

Proof ($\alpha_{k_{\max}}$)

$$\begin{pmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{pmatrix} = \sum_{k=\max(B_1,\cdots,B_4)}^{\min(A_1,\ldots,A_3)} \alpha_k$$

where

$$\alpha_k = \frac{(-1)^k \{k+1\}!}{\{1\}\{A_1-k\}!\ldots\{A_3-k\}!\{k-B_1\}!\ldots\{k-B_4\}!},$$

$$\{j\} = s_N^{2j} - s_N^{-2j} = 2\sqrt{-1}\sin\frac{2j\pi}{N}, \quad \begin{cases} \operatorname{Im}\{j\} > 0 & \text{if } 0 < j < \frac{N}{2}, \\ \operatorname{Im}\{j\} < 0 & \text{if } \frac{N}{2} < j < N. \end{cases}$$

The terms of the denominator of α_k are all positive. Hence the range of k is contained in $N/2 < k < N$ and

$$\operatorname{sign}\frac{\{k+1\}!}{\sqrt{-1}^{k+1}} = (-1)^{(N+1)/2}, \qquad \operatorname{sign}\alpha_k = (-1)^{(N+1)/2}.$$

Therefore

$$|\alpha_{k_{\max}}| \le \left|\sum_k \alpha_k\right| \le N\,|\alpha_{k_{\max}}|,$$

$$\lim_{N\to\infty}\frac{2\pi}{N}\log\left|\begin{pmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{pmatrix}\right| = \lim_{N\to\infty}\frac{2\pi}{N}\log|\alpha_{k_{\max}}|.$$

Proof ($k_{\max}$)

$$\begin{pmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{pmatrix} = \sum_{k=\max(B_1,\cdots,B_4)}^{\min(A_1,\dots,A_3)} \alpha_k$$

where

$$\alpha_k = \frac{(-1)^k \{k+1\}!}{\{1\}\{A_1-k\}! \dots \{A_3-k\}! \{k-B_1\}! \dots \{k-B_4\}!},$$

$$\{j\} = s_N^{2j} - s_N^{-2j} = 2\sqrt{-1}\sin\tfrac{2j\pi}{N}, \quad \begin{cases} \mathrm{Im}\{j\} > 0 & \text{if } 0 < j < \frac{N}{2}, \\ \mathrm{Im}\{j\} < 0 & \text{if } \frac{N}{2} < j < N. \end{cases}$$

The ratio

$$\frac{\alpha_{k+1}}{\alpha k} = -\frac{\{k+2]\{A_1-k\} \dots \{A_3-k\}}{\{k+1-B_1\} \dots \{k+1-B_4\}}$$

is positive and bigger than 1 if k is small and less than 1 if k is big, and is monotonically decreasing between $\max(B_1, B_2, B_3, B_4)$ and $\min(A_1, A_2, A_3)$. Hence α_k has a maximum value at $k_{\max}$ which satisfies $\frac{\alpha_{k_{\max}+1}}{\alpha k_{\max}} \doteqdot 1$ and so

$$-\frac{\{k_{\max}+2\}\{A_1-k_{\max}\} \dots \{A_3-k_{\max}\}}{\{k_{\max}+1-B_1\} \dots \{k_{\max}+1-B_4\}} \doteqdot 1.$$

Proof (hyperbolic volume)

Now consider about

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left| \begin{Bmatrix} a_1^{(N)} & a_2^{(N)} & a_5^{(N)} \\ a_4^{(N)} & a_3^{(N)} & a_6^{(N)} \end{Bmatrix}^{RW}_{q_N^2} \right|$$

where

$$\begin{Bmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{Bmatrix}^{RW}_{q_N^2} = \frac{\begin{pmatrix} c_1 & c_2 & c_5 \\ c_4 & c_3 & c_6 \end{pmatrix}}{\sqrt{W_2(c_1,c_2,c_5)\, W_2(c_1,c_3,c_6)\, W_2(c_2,c_4,c_6)\, W_2(c_3,c_4,c_5)}},$$

$$W_2(f) = \frac{\{\frac{c_1+c_2+c_3}{2}+1\}!}{\{1\}\,\{\frac{-c_1+c_2+c_3}{2}\}!\{\frac{c_1-c_2+c_3}{2}\}!\{\frac{c_1+c_2-c_3}{2}\}!}.$$

By using sectional mensuration, we have

$$\lim_{N\to\infty} \frac{2\pi}{N} \log|\{k\}!| = \lim_{N\to\infty} \frac{2\pi}{N} \sum_{j=1}^{k} \log\left|2\sin\frac{2\pi j}{N}\right| =$$

$$\int_0^x |2\sin t|\, dt = \Lambda(x) = \mathrm{Im}\,\mathrm{Li}_2(e^{2\sqrt{-1}x}),$$

where $x = \frac{2\pi k}{N}$. So, replace $\{k\}!$ by Li_2, then we get the volume formula for a truncated tetrahedron given by

A. Ushijima, A volume formula for generalised hyperbolic tetrahedra. *Non-Euclidean geometries*, 249–265, Math. Appl. (N. Y.), **581**, Springer, New York, 2006.

A generalizations

- **Compact tetrahedron**
  There are some oscillating terms in α_k, but they seems to be small and negligible.

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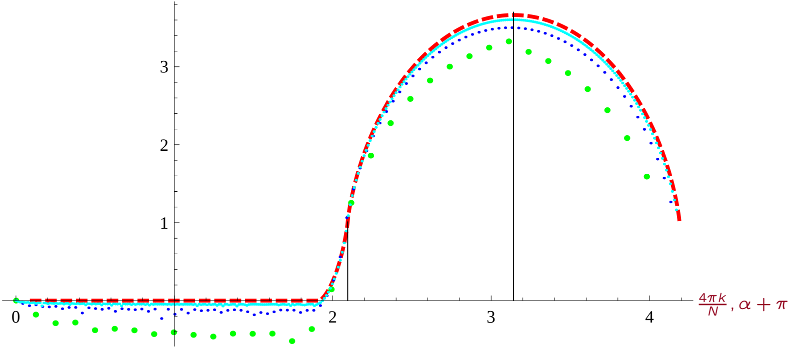
  There are some oscillating terms in α_k, but they seems to be small and negligible.



● : $\frac{2\pi}{101} \log \left| \left\{ \begin{matrix} k \dots \\ k \dots \end{matrix} \right\}^{RW}_{s=\exp \frac{2\pi i}{101}} \right|$ ● : $\frac{2\pi}{301} \log \left| \left\{ \begin{matrix} k \dots \\ k \dots \end{matrix} \right\}^{RW}_{s=\exp \frac{2\pi i}{301}} \right|$ ● : $\frac{2\pi}{1001} \log \left| \left\{ \begin{matrix} k \dots \\ k \dots \end{matrix} \right\}^{RW}_{s=\exp \frac{2\pi i}{1001}} \right|$

- - - : $\mathrm{Vol}(T_{|\pi-\alpha|})$