On an algorithm to determine the geometric structure of a 3-manifold from its simplicial decomposition

Jun Murakami[1]

Abstract. An idea to get an algorithm to determine the geometric structure of a 3-manifold is proposed.

Keyword. geometric structure of 3-manifolds.

AMS classi fications. 57M50.

1. Geometric structure determined by a volume function

Let M be a 3-manifold and $\mathcal{T}$ be a simplicial decomposition of M, and T_1, T_2, $\cdots$, T_t be the tetrahedra of $\mathcal{T}$. Parameters corresponding to the dihedral angles of the tetrahedra of $\mathcal{T}$ are given. There are $6t$ parameters. Assume that the sum of parameters around every edge of $\mathcal{T}$ is equal to 2π. Therefore, there are $6t - e$ free parameters, where e is the number of edges in $\mathcal{T}$. Let

$$P(\mathcal{T}) = \sum_{i=1}^{t} P(T_i),$$

where $P(T_i)$ is a complex number defined in the next section, whose imaginary part is equal to the volume of T_i if T_i can be realized as a hyperbolic tetrahedron.

Algorithm. Obtain a critical point of $P(\mathcal{T})$ with respect to the $6t - e$ free parameters. Then the geometric structure of each tetrahedra given by the parameters corresponding to the critical point may give the geometric structure of M.

The geometric structure of each tetrahedra is given separately. The above statement claims that the adjacent tetrahedra wrap over evenly at each adjoining edge. This claim is based on the following argument.

Let E be an edge of $\mathcal{T}$ and $T_{i(1)}$, $T_{i(2)}$, $\cdots$, $T_{i(k)}$ be the tetrahedra containing E. We parametrize the dihedral angles of these tetrahedra at E by α_1, $\alpha_2 - \alpha_1$, $\cdots$,

[1]Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama-cho 1-1, Toyonaka, Osaka 560-0043, Japan, e-mail: jun@math.sci.osaka-u.ac.jp

$\alpha_{k-1}-\alpha_{k-2}$, $2\pi-\alpha_{k-1}$. Then the equations to get the critical point of $P(\mathcal{T})$ are the following.

$$(1)\qquad \frac{P(\mathcal{T})}{d\alpha_i}=0, \qquad (i=1,2,\cdots,k-1)$$

such equations for other edges.

Let $\beta_1=\alpha_1$, $\beta_2=\alpha_2-\alpha_1$, $\cdots$, $\beta_{k-1}=\alpha_{k-1}-\alpha_{k-2}$, $\beta_k=2\pi-\alpha_{k-1}$, which present the dihedral angles at the edge E. Then (1) is equivalent to the following.

$$(2)\qquad \begin{aligned} &\frac{P(T_{i(1)})}{d\beta_1}=\frac{P(T_{i(2)})}{d\beta_2}=\cdots=\frac{P(T_{i(k)})}{d\beta_k},\\ &\beta_1+\beta_2+\cdots+\beta_k=2\pi, \end{aligned}$$

such equations for other edges.

If $T_{i(j)}$ $(j=1,\,2,\,\cdots,\,k)$ are all hyperbolic tetrahedra, Schläfli's relation

$$(3)\qquad \frac{\operatorname{Im}P(T_{i(j)})}{\beta_k}=-\frac{1}{2}\,(\text{length of } E)\quad (j=1,2,\cdots,k)$$

implies that (2) means that the lengths of the edge E obtained from the geometric structure of adjoining tetrahedra are equal, and so we can glue the structures of these tetrahedra.

Problem 1. If all the tetrahedra of $\mathcal{T}$ is realized as hyperbolic tetrahedra at the critical point, the above argument shows that the hyperbolic structure of M is actually determined. Extend this fact to the other structures.

Problem 2. The set of critical points are not discrete nor finite. How to find an appropriate critical point which is good for determine geometric structure? Does an appropriate critical point always exist?

Remark. An algorithm to determine the geometric structure from Schläfli's relation is also given by A. Casson several years ago [1]. His algorithm is to move the length of edges, and keep the total space to be a cone manifold. On the other hand, algorithm presented here is to vary the structure of each tetrahedron separately, and don't care about the structure of the total space at the beginning. The local structures eventually extend to the total space at the critical point.

2. Volume of a hyperbolic tetrahedron

A formula for the volume of a generic tetrahedron is given in [2]. Here we propose another formula, which is numerically checked for various examples.

Let T be a tetrahedron whose dihedral angles are A, B, C, D, E, F, where A, B, C correspond to the three edges containing a common vertex, and D, E, F are the angles at the opposite positions of A, B, C respectively. Let $a = \exp\sqrt{-1}(\pi - A)$, $b = \exp\sqrt{-1}(\pi - B)$, $\cdots$, $f = \exp\sqrt{-1}(\pi - F)$, and $V(z, a, b, c, d, e, f)$ be

$$V(z,a,b,c,d,e,f) = \\ \mathrm{Li}_2(z) + \mathrm{Li}_2(z\,a\,b\,d\,e) + \mathrm{Li}_2(z\,a\,c\,d\,f) + \mathrm{Li}_2(z\,b\,c\,e\,f) \\ - \mathrm{Li}_2(z\,a\,b\,c) - \mathrm{Li}_2(z\,a\,e\,f) - \mathrm{Li}_2(z\,b\,d\,f) - \mathrm{Li}_2(z\,c\,d\,e),$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function defined by the analytic continuation of the following integral.

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t},$$

where x is a real positive number.

Consider a equation

$$\frac{dV(z,a,b,c,d,e,f)}{dz} = 0. \tag{4}$$

This equation has three solutions, and one solution is the trivial one, i.e. $z = 0$. Let z_0 and z_0' be the remaining two non-trivial solution of (4). Let

$$P(T) = \frac{V(z_0,a,b,c,d,e,f) - V(z_0',a,b,c,d,e,f)}{2} \tag{5}$$

Then the following is expected.

Conjecture. (joint with M. Yano) If T is realized in a hyperbolic space,

$$\text{Volume of } T = \left|\mathrm{Im}\,P(T)\right|.$$

This formula is obtained from the quantum 6j-symbol related to the representation theory of $\mathcal{U}_q(sl_2)$. The volume conjecture of 3-manifolds suggest that certain limit of the Witten-Reshetikhin-Turaev invariant of a hyperbolic 3-manifold, say M, may relate to its hyperbolic volume. The absolute value of the square of this invariant is equal to the Turaev-Viro invariant, which comes from a simplicial decomposition of M. First, assign parameters to each edges of the decomposition, then assign 6j-symbols to each tetrahedron, and then take the state sum over all assignment of parameters, then we get the Turaev-Viro invariant of M. Therefore, it may be natural to expected that there may be some relation between the 6j-symbol and the hyperbolic volume of a hyperbolic tetrahedra. After computing several examples, we conclude that (5) may hold. This formula is proved for a ideal tetrahedron, which is a tetrahedron whose vertices are all located at the infinity.

If T is realized in a Euclidean space, $P(T)$ is equal to 0. If T is realized in 3-sphere S^3, the real part of $P(T)$ instead of the imaginary part seems to relate the volume. Therefore, the critical point of the function $P(\mathcal{T})$ of a simplicial decomposition $\mathcal{T}$ of a 3-manifold M may relates to its geometric structure.

References

[1] A. Casson, On computing geometric structures on 3-manifolds, talk at Workshop on Computational and Algorithmic Methods in Three Dimensional Topology , March 10-14, 1997, Mathematical Sciences Research Institute, *etc.*

[2] Y. Cho and H. Kim, *On the volume formula for hyperbolic tetrahedra,* Discrete Comput. Geom. **22** (1999), 347–366.