REPRESENTATION OF MAPPING CLASS GROUPS VIA THE UNIVERSAL PERTURBATIVE INVARIANT

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ABSTRACT. We introduce a new method to construct representations of mapping class groups of closed oriented surfaces by using the universal perturbative invariant of 3-manifolds with boundary. We also give examples for genus one and two cases, including a representation which is non-trivial on the Torelli subgroup.

INTRODUCTION.

By using the Kontsevich invariant [4, 1], Le, Ohtsuki and the author introduced in [5] the universal perturbative invariant of closed 3-manifold. It is $\mathcal{A}(\phi)$-valued invariant, where $\mathcal{A}(\phi)$ is a C-linear span of trivalent graphs with relations "anti-symmetry relation" around the vertex and "IHX relation" corresponding to the Jacobi identity of Lie algebras. $\mathcal{A}(\phi)$ has an algebra structure with multiplication given by the disjoint union of two graphs.

In [7], this invariant is generalized for 3-manifolds with boundary. Let M be a 3-manifold with boundary ∂M. If the boundary ∂M is connected and of genus g, then the values of this invariant is in the $\mathcal{A}(\phi)$-module $\mathcal{A}(\Gamma_g)$, where $\mathcal{A}(\Gamma_g)$ is the space of chord diagrams on the graph Γ_g, and Γ_g is the oriented trivalent graph with g circles as in Figure 1. We call Γ_g the *chain graph* of genus g. By considering the action of the mapping class group of ∂M, we derive representation of the mapping class group of a closed surface of genus g.

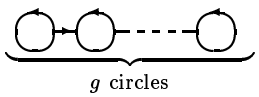

FIGURE 1. The graph Γ_g corresponding to a closed surface of genus g

Key words and phrases. mapping class groups, invariant of 3-manifolds, topological quantum field theory.

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1. Universal perturbative invariant of graphs embedded in 3-manifolds

In this section, we review the result in Section 2 of [7]. Some parts are modified for later use.

1.1. **Space of chord diagrams.** Let X be a union of copies of S^1 and framed trivalent graphs. A chord diagram with support X means a set of pairs of distinct points on X. Graphically, each pair is displayed by a dashed line, and the two end points of the dashed line represent the corresponding pair of points. Let $\mathcal{A}(X)$ be the C-linear space spanned by chord diagrams with support X, subject to the AS, IHX and STU relation in [7]. By STU relation, we impose chord diagrams with trivalent vertices of dashed lines. We call $\mathcal{A}(X)$ the *space of chord diagrams with support* X.

Let D be a chord diagram in $\mathcal{A}(X)$. A vertex of D is called an *inner vertex* if it is end points of only dashed lines. We define the *i-degree* of D to be the number of inner vertices of D. Let $\mathcal{A}_n(X)$ be the subspace of $\mathcal{A}(X)$ spanned by chord diagrams with i-degree more or equal to n. Then $\mathcal{A}(X)$ is filtered as

$$\mathcal{A}(X) = \mathcal{A}_0(X) \supset \mathcal{A}_1(X) \supset \mathcal{A}_2(X) \supset \cdots .$$

1.2. **Invariant of graphs embedded in a 3-manifold.** Let L be a framed link and G an embedding of Γ in S^3 disjoint to L as before. Let M be a 3-manifold obtained by the surgery along L. Then, after the surgery, the image of G is an embedding of Γ in M, and we use G again for this embedding in M. Let $\mathring{\mathcal{A}}(X)$ be the space of chord diagrams including trivial circles of dashed lines. Let P_n be the relation in Figure 2 introduced in [5]. Let O_n be the relation also introduced in [5] so that a trivial circle of a dashed line is equivalent to $-2n$. Let

$$\mathcal{A}^{(n)}(\Gamma) = \mathcal{A}(\Gamma)/O_n, P_{n+1}, \mathcal{A}_{2n+1}(\Gamma)$$

Then $\mathcal{A}^{(n)}(\Gamma)$ is a $\mathcal{A}^{(n)}(\phi)$-module where the multiplication is given by the split union of chord diagrams. We call n the *degree*. As in [7], we have a natural coproduct

$$\Delta_{n,m} : \mathcal{A}^{(n)}(\Gamma) \longrightarrow \mathcal{A}^{(m)}(\Gamma) \otimes \mathcal{A}^{(n-m)}(\Gamma),$$

which is a $\mathcal{A}^{(n)}(\phi)$-module homomorphism.

Let $\iota^{(n)}$ the mapping from $\mathcal{A}(\sqcup^{\ell} S^1 \cup \Gamma)$ to $\mathcal{A}^{(n)}(\Gamma)$ defined in [7] by replacing the solid lines with m end points of dashed lines by T^n_m and then take the image in $\mathcal{A}^{(n)}(\Gamma)$. Recall that $T^n_m = 0$ if $m < 2n$.

Let $\check{Z}(L \cup G)$ be the invariant of $L \cup G$ in S^3 defined in Section 2.2 of [7], which is in $\mathcal{A}(\sqcup^\ell S^1 \cup \Gamma)$, where ℓ is the number of connected components of L. Let $\check{Z}^{(n)}(L \cup G)$ be the image of $\check{Z}(L \cup G)$ in $\mathcal{A}^{(n)}(\sqcup^\ell S^1 \cup \Gamma)$.

Let U_+ and U_- be trivial knots with ± 1 framings respectively. Let

$$Y^{(n)}(L \cup G) = \frac{\iota^{(n)}(\check{Z}(L \cup G))}{\left(\sqrt{\iota^{(n)}(\check{Z}(U_+))\,\iota^{(n)}(\check{Z}(U_-))}\right)^\ell},$$

where ℓ is the number of components of L. The invariant $\Omega_n(M, G)$ in [7] is normalized by signatures concerning to M and G, while $Y^{(n)}(L \cup G)$ in $\mathcal{A}^{(n)}(\Gamma)$ is normalized only for the number of components. Note also that we impose different degree to the space $\mathcal{A}^{(n)}(\Gamma)$ from the degree of $\mathcal{A}(\Gamma)$ in [7].

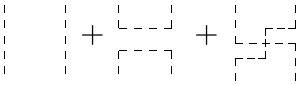
P_1 : P_2 :  P_3 : $\cdots\cdots$

FIGURE 2. The reltaion P_n

2. Representations of mapping class groups

2.1. **Action of the mapping class group $\mathcal{M}_g$.** Let $\mathcal{M}_g$ be the mapping class group of a closed oriented surface of genus g. Then $\mathcal{M}_1$ is generated by α_1 and β_1, and $\mathcal{M}_g$ is generated by α_1, β_1, α_2, β_2, $\cdots$, α_g, β_g and δ in Figure 3 as shown in [2]. A simple relation for these generators is given in [8].

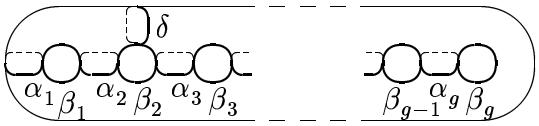


FIGURE 3. Generators of $\mathcal{M}_g$

Let M be a 3-manifold whose boundary is a genus g surface S. In S, a set F of meridians and framings are fixed. Filling the boundary S of M by the handle body of genus g along the meridians and framings, we get a closed 3-manifold $\widehat{M}$. Let G be the embedding of the chain graph Γ_g in the handle body as in Figure 4. Let e_1, e_2, $\cdots$, e_{3g-3} be the edges of G as in Figure 4.

The mapping class group $\mathcal{M}_g$ acts on S, and this action changes the set of meridians and framings of S. Let x be an element of $\mathcal{M}_g$, and $(\widehat{xM}, xG)$ the pair of the 3-manifold

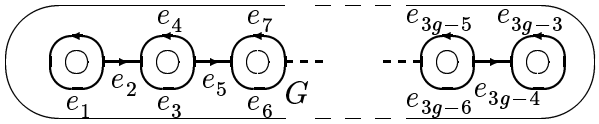


FIGURE 4. A chain graph embedded in a handle body

and the graph embedded in $\widehat{xM}$ corresponding to xF, the set of meridians and framings of S changed by the action of x.

2.2. **Surgery descriptions of the action.** The actions of the generators of $\mathcal{M}_g$ are described by surgeries. The actions of α_1, α_g and δ are given by the -1 surgery along a trivial knot around the edge e_1, e_4 and e_{3g-3} respectively as in Figure 5. The action of α_i $(2 \leq i \leq g-1)$ is given by the -1 surgery along a trivial knot around the edges e_{3i-5} and e_{3i-3} as in Figure 5.

The action of β_i is given as follows. Make a knot parallel to the edges e_{3i-3} and e_{3i-2} as in Figure 5 and add -1 to the framing of this newly added knot. Then the surgery along this knot gives the action of β_i.

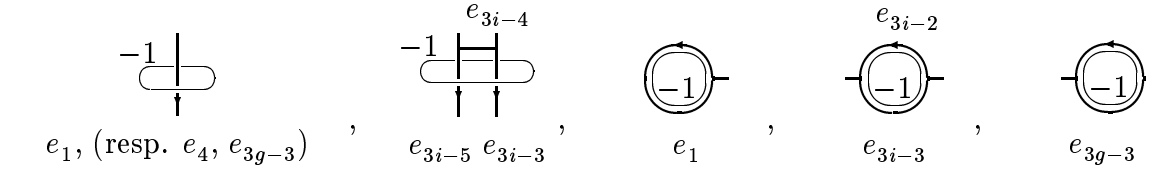


FIGURE 5. Actions of the generators

By changing the numbers -1 for the framings to $+1$ in the above constructions, we get actions of inverses of the corresponding generators.

2.3. **Central extension of $\mathcal{M}_g$.** Assume that $\widehat{M}$ is obtained by a surgery along a framed link L in S^3 whose image is disjoint with the graph G. We also denote the preimage of G in S^3 by G. From $L \cup G$ in S^3, we defined $Y^{(n)}(L \cup G)$ in $\mathcal{A}^{(n)}(\Gamma_g)$.

Let w be a word of generators of $\mathcal{M}_g$. By using the surgery description of the actions, we define wL as a union of L and all knots added by the actions of generators in w. If the word w represents the identity in $\mathcal{M}_g$, then, by using the Kirby moves for a graph embedded in a 3-manifold introduced in [7], wL is equivalent to a split union $L \sqcup L'$, where L' is a split union of trivial knots with ± 1 framings. Let $s(w)$ be the sum of framings of

L'. Let $\widetilde{\mathcal{M}}_g$ be a central extension of $\mathcal{M}_g$ corresponding to the signature, i.e.

$$\widetilde{\mathcal{M}}_g = < {\alpha_1}^{\pm 1}, \cdots, {\alpha_g}^{\pm 1}, \delta^{\pm 1}, {\beta_1}^{\pm 1}, \cdots, {\beta_g}^{\pm 1}, \gamma^{\pm 1} \mid \gamma : \text{central element},$$
$$w = \gamma^{s(w)} \quad \text{if a word } w \text{ represents identity in } \mathcal{M}_g > .$$

Let $\widehat{wM}$ be the 3-manifold obtained by the surgery along wL. Then (wL, G) corresponds to $(\widehat{wM}, G)$. Let $wY^{(n)}(L \cup G) = Y^{(n)}(wL \cup G)$, then this gives a well-defined action of $\tilde{c}M_g$ to $Y^{(n)}(L \cup G)$.

2.4. **Action on $\mathcal{A}^{(n)}(\Gamma_g)$.** We extend the previous action of $\widetilde{\mathcal{M}}_g$ to the action on $\mathcal{A}(\Gamma_g)$. Let ε_i be the edge of Γ corresponding to the edge e_i of G. Let H be a (1, 1)-tangle with a straight string and a -1 framed trivial knot around the string as in Figure 6. Let

$$\eta = \sqrt{\iota^{(n)}(\check{Z}(U_+))\, \iota^{(n)}(\check{Z}(U_-))}.$$

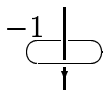


FIGURE 6. The tangle associated with the action of α_1

Let x be an element in $\mathcal{A}(\Gamma_g)$. We assume that end points of chords in the chord diagrams in x are only on the edges ε_1, ε_4, ε_7, $\cdots$, ε_{3g-8}, ε_{3g-5}, ε_{3g-3}. The actions of α_1, α_g and δ are defined by

$$\alpha_1 x = \eta^{-1}\, \iota^{(n)}(\hat{Z}(H) \underset{\varepsilon_1}{\#} x), \quad \delta x = \eta^{-1}\, \iota^{(n)}(\hat{Z}(H) \underset{\varepsilon_4}{\#} x), \quad \alpha_g x = \eta^{-1}\, \iota^{(n)}(\hat{Z}(H) \underset{\varepsilon_{3g-3}}{\#} x).$$

where

$$\nu = Z(\bigcirc)^{-1},$$

the Kontsevich invariant of a 0-framed knot $\bigcirc$, which is a normalization factor used in the definition of link invariant $\hat{Z}$, and $\underset{\varepsilon_i}{\#}$ means the insertion of $\hat{Z}(H)$ to the edge ε_i. By using the second Kirby move, inserting H to an edge ε of the graph G is equivalent to insert a positive twist at the edge ε and add a trivial -1 framed knot U_- as a split union as in Figure 7. Hence, we get the following.

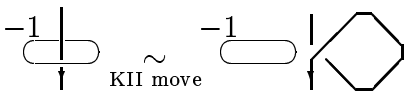


FIGURE 7. Apply the second Kirby move (KII) to H

Lemma 2.1. *The action of α_1, δ and α_g are given by*

$$\alpha_1 x = \eta^{-1}\,\iota^{(n)}(\check{Z}(U_-))\,(\exp(\frac{\zeta}{2})\underset{\varepsilon_1}{\#}x), \qquad \delta x = \eta^{-1}\,\iota^{(n)}(\check{Z}(U_-))\,(\exp(\frac{\zeta}{2})\underset{\varepsilon_4}{\#}x),$$

$$\alpha_g x = \eta^{-1}\,\iota^{(n)}(\check{Z}(U_-))\,(\exp(\frac{\zeta}{2})\underset{\varepsilon_{3g-3}}{\#}x).$$

where ζ is the chord diagram on one open solid line with one chord as in Figure 8, and

$$\exp(\frac{\zeta}{2}) = (\text{open solid line without chord}) + \sum_{k=1}^{\infty}\frac{\zeta^k}{2^k\,k!} \tag{2.1}$$

FIGURE 8. Chord diagram ζ

The actions of α_i is defined by

$$\alpha_i x = \eta^{-1}\,\iota^{(n)}(S_1(\Delta(\widehat{Z}(H)))\underset{\varepsilon_{3i-5},\varepsilon_{3i-3}}{\#}x),$$

where Δ is the coproduct applied to the straight string of H, S_1 is the antipode for the left string of $\Delta(\widehat{Z}(H))$, and $\underset{\varepsilon_{3i-5},\varepsilon_{3i-3}}{\#}$ means the substitution of $\Delta(\widehat{Z}(H))$ just below the edge ε_{3i-4} as in Figure 9. By using the second Kirby move as before, we get the following.

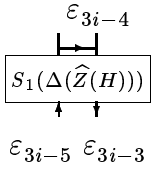


FIGURE 9. Action of α_i

Lemma 2.2. *The action of α_i is given by*

$$\alpha_i x = \eta^{-1}\,\iota^{(n)}(\check{Z}(U_-))\,(S_1(\Delta(\exp(\frac{\zeta}{2})))\underset{\varepsilon_{3i-5},\varepsilon_{3i-3}}{\#}x).$$

The action of β_i is given by

$$\beta_i x = \eta^{-1}\,\iota^{(n)}(x_i),$$

where x_i is given in Figure 10. In the figure, Δ means the coproduct acting on the end points on the edges.

From the construction of the above actions, we get the following.

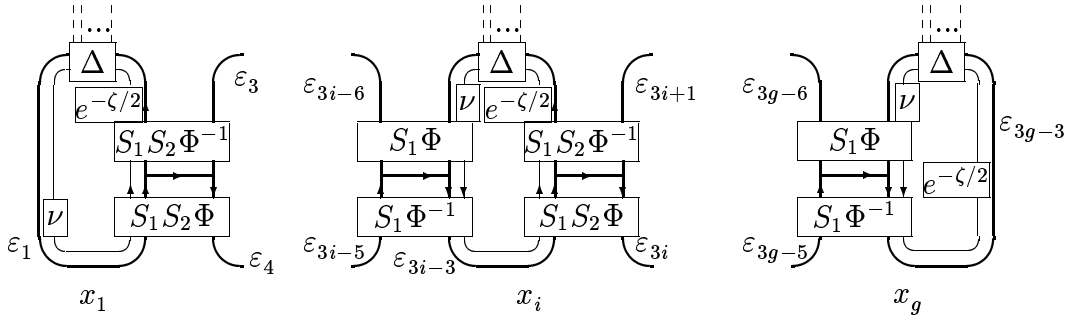


FIGURE 10. x_i

Proposition 2.3. *The actions of $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, \delta$ defines an action of $\widetilde{\mathcal{M}}_g$ on $\mathcal{A}^{(n)}(\Gamma_g)$. The central element γ acts as a multiplication by $\eta^{-1}\,\iota^{(n)}(\check{Z}(U_+))$ in $\mathcal{A}^{(n)}(\phi)$.*

Let $\widetilde{\rho}_g^{(n)}$ denote the above representation of $\widetilde{\mathcal{M}}_g$ on $\mathcal{A}^{(n)}(\Gamma_g)$. We call $\widetilde{\rho}_g^{(n)}$ *a perturbative representation of genus g and degree n.*

Remark . Let $\rho_g^{(n)}$ be the representation of $\widetilde{\mathcal{M}}_g$ defined by

$$\rho_g^{(n)}(x) = \det(\widetilde{\rho}_g^{(n)}(x))^{-1}\,\widetilde{\rho}_g^{(n)}(x).$$

If $g = 1$ or 2, then $\widetilde{\mathcal{M}}_g$ is isomorphic to $\mathcal{M}_g \times \mathrm{Z}$. Hence, $\rho_g^{(n)}$ gives a representation of $\mathcal{M}_g$.

3. Perturbative representation of genus one and degree one

In the rest of this note, we explicitly describe representations $\widetilde{\rho}_g^{(n)}$ of $\widetilde{\mathcal{M}}_g$ on $\mathcal{A}^{(n)}(\Gamma_g)$ for some pairs of g and n. For the genus one and degree one case, such representation is also constructed in [6] with slightly different action.

In the rest of this paper, θ denote the chord diagram in $\mathcal{A}(\phi)$ of shape like the letter θ, i.e.

$$\theta = \quad . \tag{3.1}$$

3.1. Preparation.

The algebra $\mathcal{A}^{(1)}(\phi)$ is a two-dimensional algebra over C, i.e.

$$\mathcal{A}^{(1)}(\phi) \cong \mathrm{C} + \mathrm{C}\,\theta,$$

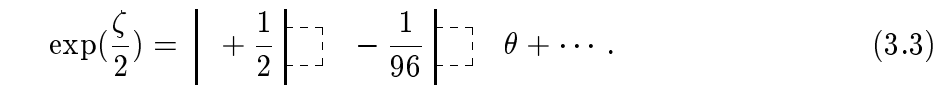
where $\theta^2 = 0$. We already know $\iota^{(1)}(U_\pm)$ as in Lemma 4.3 of [5], which is

$$\iota^{(1)}(\check{Z}(U_\pm)) = \mp 1 + \frac{\theta}{16}.$$

Hence, $\eta = \sqrt{-1}$, and so $\eta^{-1}\iota^{(1)}(\check{Z}(U_-)) = -\sqrt{-1}\,(1+\frac{\theta}{16})$. The space $\mathcal{A}^{(1)}(S^1)$ is spanned by and as $\mathcal{A}^{(1)}(\phi)$-module. Modulo the relation P_2, we have

$$= -\frac{1}{12}\ \theta \tag{3.2}$$

and

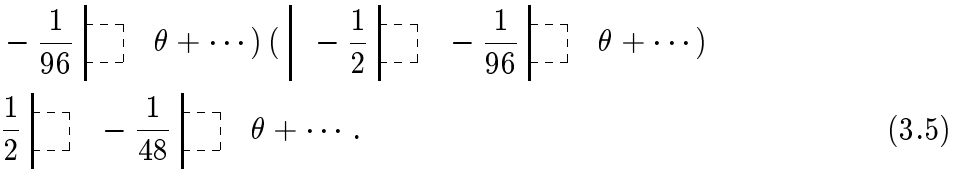


$$\exp(\frac{\zeta}{2}) = \ + \frac{1}{2}\ - \frac{1}{96}\ \theta + \cdots . \tag{3.3}$$

3.2. **Action of α_1.** The action of the generator α_1 is given by inserting $\exp(\frac{\zeta}{2})$ and multiplying $\eta^{-1}\iota^{(1)}(\check{Z}(U_-))$. Therefore, the representation matrix of $\tilde{\rho}_1^{(1)}(\alpha_1)$ with respect to the basis and is given by

$$\tilde{\rho}_1^{(1)}(\alpha_1) = -\sqrt{-1}\,(1+\frac{\theta}{16}) \begin{pmatrix} 1 & 0 \\ \frac{1}{2} - \frac{1}{96}\theta & 1 - \frac{1}{24}\theta \end{pmatrix}. \tag{3.4}$$

3.3. **Action of β_1.** The action of β_1 is to apply Δ, insert $\nu\exp(-\frac{\zeta}{2})$ and apply $\iota^{(1)}$ to the newly added component, and then multiply by η^{-1}. Modulo P_2 relation, we have



$$\begin{aligned} \nu\exp(-\frac{\zeta}{2}) &= (\ - \frac{1}{96}\ \theta + \cdots)\,(\ - \frac{1}{2}\ - \frac{1}{96}\ \theta + \cdots) \\ &= \ - \frac{1}{2}\ - \frac{1}{48}\ \theta + \cdots . \end{aligned} \tag{3.5}$$

Hence, we have

$$\tilde{\rho}_1^{(1)}(\beta_1) = -\sqrt{-1} \begin{pmatrix} 1 + \frac{1}{24}\theta & -2 - \frac{1}{12}\theta \\ 0 & 1 + \frac{1}{24}\theta \end{pmatrix}. \tag{3.6}$$

4. Perturbative representation of genus two and degree one

4.1. **Preparation.** The space $\mathcal{A}^{(1)}(\Gamma_2)$ is spanned by the following 5 chord diagrams C_1, C_2, C_3, C_4, C_5 in Figure 3 as a $\mathcal{A}^{(1)}(\phi)$-module.

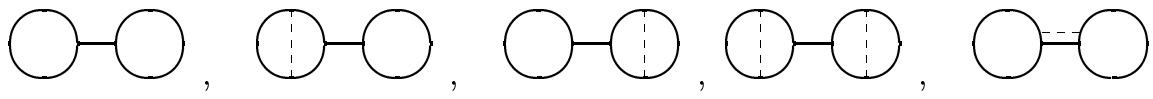
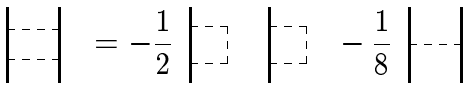


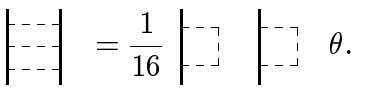
Figure 11. Generators of $\mathcal{A}^{(1)}(\Gamma_2)$

We prepare some formulas modulo the P_2 relation. First, we have



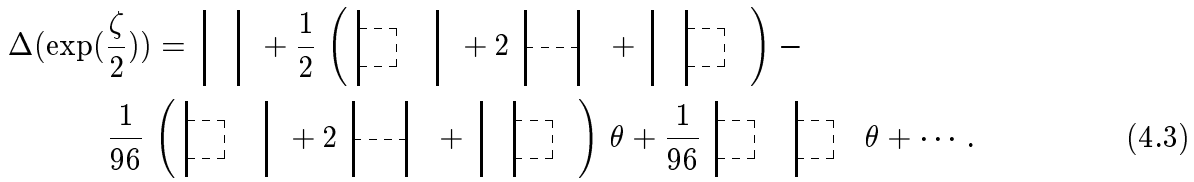
(4.1)

From this formula, we get



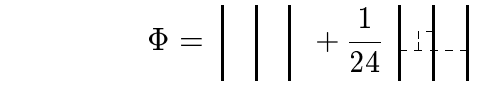
(4.2)

For $\exp(\zeta/2)$, we have



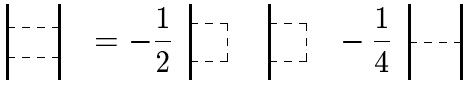
(4.3)

We know that



(4.4)

Moreover, we have



(4.5)

4.2. **Action of α_1.** As in the genus one case, the action of α_1 is to insert $\exp(\frac{\zeta}{2})$ to the edge ε_1 and then multiply by $\eta^{-1}\iota^{(1)}(\check{Z}(U_-))$. We have

$$\exp(\frac{\zeta}{2})\,C_1 = C_1 + \left(\frac{1}{2} - \frac{\theta}{96}\right) C_2, \quad \exp(\frac{\zeta}{2})\,C_2 = \left(1 - \frac{\theta}{24}\right) C_2,$$

$$\exp(\frac{\zeta}{2})\,C_3 = C_3 + \left(\frac{1}{2} - \frac{\theta}{96}\right) C_4, \quad \exp(\frac{\zeta}{2})\,C_4 = \left(1 - \frac{\theta}{24}\right) C_4,$$

$$\exp(\frac{\zeta}{2})\,C_5 = \left(1 - \frac{\theta}{24}\right) C_5.$$

Hence the representation matrix of α_1 with respect to the basis in Figure 11 is given by

$$\tilde{\rho}_2^{(1)}(\alpha_1) = -\sqrt{-1}\,(1+\frac{\theta}{16}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} - \frac{\theta}{96} & 1 - \frac{\theta}{24} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{\theta}{96} & 1 - \frac{\theta}{24} & 0 \\ 0 & 0 & 0 & 0 & 1 - \frac{\theta}{24} \end{pmatrix}. \tag{4.6}$$

4.3. **Action of α_2.** The action of α_2 is to insert $\Delta(\exp(\frac{\zeta}{2}))$ to the edges ε_1 and ε_3 just below ε_2 and multiply $\eta^{-1}\iota^{(1)}(\check{Z}(U_-))$. By using (4.3), insertion of $\Delta(\exp(\frac{\zeta}{2}))$ to each generator is given as follows.

$$\Delta(\exp(\frac{\zeta}{2}))\, C_1 = C_1 + \left(\frac{1}{2} - \frac{\theta}{96}\right) C_2 + \left(\frac{1}{2} - \frac{\theta}{96}\right) C_3 + \left(-1 + \frac{\theta}{48}\right) C_5,$$

$$\Delta(\exp(\frac{\zeta}{2}))\, C_2 = \left(1 - \frac{\theta}{24}\right) C_2 + \left(\frac{1}{2} - \frac{\theta}{96}\right) C_4 + \frac{\theta}{12} C_5,$$

$$\Delta(\exp(\frac{\zeta}{2}))\, C_3 = \left(1 - \frac{\theta}{24}\right) C_3 + \left(\frac{1}{2} - \frac{\theta}{96}\right) C_4 + \frac{\theta}{12} C_5,$$

$$\Delta(\exp(\frac{\zeta}{2}))\, C_4 = \left(1 - \frac{\theta}{12}\right) C_4,$$

$$\Delta(\exp(\frac{\zeta}{2}))\, C_5 = \left(-\frac{1}{2} - \frac{\theta}{96}\right) C_4 + \left(1 + \frac{\theta}{24}\right) C_5.$$

Hence, the representation matrix of α_2 is given by

$$\widetilde{\rho}_2^{(1)}(\alpha_2) = -\sqrt{-1}\,(1 + \frac{\theta}{16}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} - \frac{\theta}{96} & 1 - \frac{\theta}{24} & 0 & 0 & 0 \\ \frac{1}{2} - \frac{\theta}{96} & 0 & 1 - \frac{\theta}{24} & 0 & 0 \\ 0 & \frac{1}{2} - \frac{\theta}{96} & \frac{1}{2} - \frac{\theta}{96} & 1 - \frac{\theta}{12} & \frac{1}{2} - \frac{\theta}{96} \\ -1 + \frac{\theta}{48} & \frac{\theta}{12} & \frac{\theta}{12} & 0 & 1 + \frac{\theta}{24} \end{pmatrix}. \tag{4.7}$$

4.4. **Action of β_i.** The action of β_1 is to make parallel of ε_1, insert the associator Φ, $\exp(-\frac{\zeta}{2})$ and ν as indicated in Figure 10, apply $\iota^{(1)}$ to the newly added component, and

then multiply η^{-1}. The action of Δ_{ε_1}, Φ, $\exp(-\frac{\zeta}{2})$, ν and $\iota^{(1)}$ is given as follows.

$$\iota^{(1)}\,\nu\,\exp(-\frac{\zeta}{2})\,\Phi\,\Delta_{\varepsilon_1}\,C_1 = \left(1+\frac{1}{24}\,\theta\right)\,C_1,$$

$$\iota^{(1)}\,\nu\,\exp(-\frac{\zeta}{2})\,\Phi\,\Delta_{\varepsilon_1}\,C_2 = \left(-2-\frac{\theta}{12}\right)\,C_1+\left(1+\frac{\theta}{24}\right)\,C_2-\frac{\theta}{24}\,C_5,$$

$$\iota^{(1)}\,\nu\,\exp(-\frac{\zeta}{2})\,\Phi\,\Delta_{\varepsilon_1}\,C_3 = \left(1+\frac{1}{24}\,\theta\right)\,C_3,$$

$$\iota^{(1)}\,\nu\,\exp(-\frac{\zeta}{2})\,\Phi\,\Delta_{\varepsilon_1}\,C_4 = \left(-2-\frac{1}{12}\,\theta\right)\,C_3+\left(1+\frac{1}{24}\,\theta\right)\,C_4,$$

$$\iota^{(1)}\,\nu\,\exp(-\frac{\zeta}{2})\,\Phi\,\Delta_{\varepsilon_1}\,C_5 = \left(1+\frac{1}{48}\,\theta\right)\,C_5.$$

Hence, the representation matrix of β_1 is given by

$$\widetilde{\rho}_2^{(1)}(\beta_1) = -\sqrt{-1}\begin{pmatrix} 1+\frac{\theta}{24} & -2-\frac{\theta}{12} & 0 & 0 & 0 \\ 0 & 1+\frac{\theta}{24} & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{\theta}{24} & -2-\frac{\theta}{12} & 0 \\ 0 & 0 & 0 & 1+\frac{\theta}{24} & 0 \\ 0 & -\frac{\theta}{24} & 0 & 0 & 1+\frac{\theta}{48} \end{pmatrix}. \qquad (4.8)$$

Similarly, the representation matrix of β_2 is given by

$$\tilde{\rho}_2^{(1)}(\beta_2) = -\sqrt{-1}\begin{pmatrix} 1+\frac{\theta}{24} & 0 & -2-\frac{\theta}{12} & 0 & 0 \\ 0 & 1+\frac{\theta}{24} & 0 & -2-\frac{\theta}{12} & 0 \\ 0 & 0 & 1+\frac{\theta}{24} & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{\theta}{24} & 0 \\ 0 & 0 & -\frac{\theta}{24} & 0 & 1+\frac{\theta}{48} \end{pmatrix}. \tag{4.9}$$

4.5. **Action of δ.** As the action of α_1, the action of δ is to insert $\exp(\frac{\zeta}{2})$ to the edge ε_3 and then multiply $\eta^{-1}\iota^{(1)}(\check{Z}(U_-))$. Hence the representation matrix of δ is given by

$$\tilde{\rho}_2^{(1)}(\delta) = -\sqrt{-1}\,(1+\frac{\theta}{16})\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}-\frac{\theta}{96} & 0 & 1-\frac{\theta}{24} & 0 & 0 \\ 0 & \frac{1}{2}-\frac{\theta}{96} & 0 & 1-\frac{\theta}{24} & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{\theta}{24} \end{pmatrix}. \tag{4.10}$$

4.6. **Jones representation.** A 5-dimensional representation of $\widetilde{\mathcal{M}}_2$ is also given in [3]. We denote this representation by π. Then the representation matrices of π is given as

follows.

$$\pi(\alpha_1)=\begin{pmatrix}-1&0&0&0&q\\0&-1&1&0&0\\0&0&q&0&0\\0&0&1&-1&0\\0&0&0&0&q\end{pmatrix},\qquad \pi(\beta_1)=\begin{pmatrix}-1&0&0&q&0\\0&-1&1&0&0\\0&0&q&0&0\\0&0&0&q&0\\0&0&1&0&-1\end{pmatrix},$$

$$\pi(\alpha_2)=\begin{pmatrix}q&0&0&0&0\\1&-1&0&0&0\\0&0&-1&0&q\\1&0&0&-1&0\\0&0&0&0&q\end{pmatrix},\qquad \pi(\beta_2)=\begin{pmatrix}-1&q&0&0&q\\0&q&0&0&0\\0&0&q&0&0\\0&0&1&-1&0\\0&0&1&0&-1\end{pmatrix},$$

$$\pi(\delta)=\begin{pmatrix}-1&0&0&0&q\\0&-1&1&0&0\\0&0&q&0&0\\0&0&1&-1&0\\0&0&0&0&q\end{pmatrix}.$$

The representation $\widetilde{\rho}_2^{(1)}$ at $\theta=0$ is equivalent to $\sqrt{-1}\,\pi$ at $q=-1$. The equivalence is given by the following base change.

$$(C_1,C_2,C_3,C_4,C_5)\longrightarrow(-\frac{1}{2}\,C_4,-C_2+C_5,2\,C_1,C_5,-C_3-C_5). \tag{4.11}$$

4.7. **Restriction to the Torelli subgroup.** The Torelli subgroup of $\mathcal{M}_2$ is generated by $t=(\alpha_1\beta_1\alpha_1)^4$. The representation matrix of t is the following.

$$\widetilde{\rho}_2^{(1)}(t)=\begin{pmatrix}1+\frac{\theta}{2}&0&0&0&0\\0&1+\frac{\theta}{2}&0&0&0\\0&0&1+\frac{\theta}{2}&0&0\\0&0&0&1+\frac{\theta}{2}&0\\0&0&0&0&1+\frac{\theta}{4}\end{pmatrix}. \tag{4.12}$$

This matrix is not a scalar matrix and so the representation matrix $\rho_2^{(1)}(t)$ is not the identity matrix. Hence, we have the following.

Proposition 4.1. *The restriction of the representation $\rho_2^{(1)}$ to the Torelli subgroup of $\mathcal{M}_2$ is a non-trivial representation.*

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