Actual computation for the complexified hyperbolic volume conjecture

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Absract. In this report, some computations to check the complexified hyperbolic volume conjecture in [4] are explained. This conjecture concerns the first term of the asymptotics of $\lim_{N\to\infty} J_N(K)$ of a hyperbolic knot K. In the workshop at IIAS, Hikami explained his observation about the second term of the asymptotics of $\lim_{N\to\infty} J_N(K)$, which is also positively cheched in the computation of this report.

Acknowlagement. I would like to thank all participants of the workshop at IIAS for useful discussion, especially K. Hikami for introducing his technique to do actual computation by using Pari-Gp.

1. Introduction.

1.1. **Complexified hyperbolic volume conjecture.** For a hyperbolic knot K in S^3, let $\mathrm{Vol}(K)$ and $\mathrm{CS}(K)$ be the hyperbolic volume and the Chern-Simons invariant respectively of the complement of K. Then the *complexified hyperbolic volume conjecture* [4] is the following formula to explain $\mathrm{Vol}(K)$ and $\mathrm{CS}(K)$ as a certain limit of the colored Jones invariants.

(1) $\boxed{\textbf{Complexified hyperbolic volume conjecture}}$ $$\lim_{N\to\infty} J_N(K) = \exp\left(\frac{N}{2\pi}\left(\mathrm{Vol}(K) + \sqrt{-1}\ \mathrm{CS}(K)\right)\right)$$

Here $J_N(K)$ is the colored Jones polynomial corresponding to the N dimensional representation $\mathcal{U}_q(sl_2)$ with the parameter q specialized to $\exp(2\pi\sqrt{-1}/N)$, the primitive N-th root of unity. This invariant $J_N(K)$ is proved in [3] to be equal to the Kashaev's invariant $K_N(K)$. The exact meaning of the imaginary part is given in (3).

This congecture is based on the following Kashaev conjecture. In [1], Kashaev conjectured that

$$(2) \qquad \boxed{\textbf{Kashaev conjecture}} \qquad \lim_{N\to\infty} |K_N(K)| = \exp \frac{N\,\mathrm{Vol}(K)}{2\pi}.$$

He checked this relation exactly for the figure-eight knot 4_1 and numerically for the knots 5_2 and 6_1.

The arguments of $K_N(K)$ $(= J_N(K))$ are investigated in [4] and the complexified hyperbolic volume conjecture (1) is proposed. To give the exact meaning of the imaginary part of (1), it may be better to consider the following relation:

$$(3) \qquad \lim_{N\to\infty} \frac{J_{N+1}(K)}{J_N(K)} = \exp\left(\frac{1}{2\pi}\left(\mathrm{Vol}(K) + \sqrt{-1}\ \mathrm{CS}(K)\right)\right),$$

which is cheched numerically for some examples in the rest of this report.

Kashaev conjectured (2) for hyperbolic knots, and it is generalized in [3] for any knot K as follows:

$$(4) \qquad \boxed{\text{Volume conjecture}} \qquad |J_N(K)| \underset{N\to\infty}{\sim} \exp\left(N\,\frac{v_3\,|S^3 \setminus K|}{2\,\pi}\right),$$

where $|S^3 \setminus K|$ denotes Gromov's simplicial volume of $S^3 \setminus K$, and v_3 is the volume of the ideal regular tetrahedron of the hyperbolic 3-space H^3, i.e.

$$v_3 = 1.01494160640965362502120 2554$$

It may be natural to consider about a complexification of the volume conjecture, which might be the following form.

(5)

$$\boxed{\text{Complexified volume conjedture}} \quad J_N(K) \underset{N\to\infty}{\sim} \exp\left(N\left(\frac{v_3\,|S^3 \setminus K|}{2\,\pi} + \sqrt{-1}\ \mathrm{CS}(K)\right)\right),$$

or, its quotient version

$$(6) \qquad \frac{J_{N+1}(K)}{J_N(K)} \underset{N\to\infty}{\sim} \exp\left(N\left(\frac{v_3\,|S^3 \setminus K|}{2\,\pi} + \sqrt{-1}\ \mathrm{CS}(K)\right)\right).$$

2. Hikami's observation.

Hikami observed that

$$2\,\pi \log |J_N(K)| \underset{N\to\infty}{\sim} \mathrm{Vol}(K)\,N + 3\,\pi\,\log N + O\left(\frac{1}{N}\right) \tag{7}$$

(7) $\boxed{\text{Hikami's observation}}$

for several hyperbolic prime knots. The volume $\mathrm{Vol}(K)$ in the first term corresponds to the Kashaev's conjecture (2). The second term explains a misterious behavior of $|J_N(K)|$, since the coefficient 3π appears for every prime knot he checked. Moreover, Kashaev and Tirkkonen [2] proved that

$$|J_N(K)| \sim N^{3/2} \tag{8}$$

for any torus knot K. This implies that

$$2\,\pi\,\log |J_N(K)| \sim 3\,\pi\,\log N. \tag{9}$$

Now reformulate (7) for $J_{N+1}(K)/J_N(K)$ to compare the complexified hyperbolic volume conjecture. Since

$$\log(N+1) - \log N \underset{N\to\infty}{\sim} \frac{1}{N},$$

Hikami's observation (7) is reformulated as follows.

$$2\,\pi\,\log \left|\frac{J_{N+1}(K)}{J_N(K)}\right| \underset{N\to\infty}{\sim} \mathrm{Vol}(K) + \frac{3\,\pi}{N} + O\left(\frac{1}{N^2}\right). \tag{10}$$

This relation seems to be true for all the examples given in this report.

3. Actual computations for several knots

3.1. Preliminaries. Let N be a positive integer and

$$q = \exp 2\pi\sqrt{-1}/N.$$

Let

$$(x)_k = \prod_{i=1}^{k}(1 - x^i).$$

It is known that

$$(q)_{N-1} = (\bar{q})_{N-1} = N,$$

and so

$$\frac{1}{(q)_i} = \frac{(\bar{q})_{N-1-i}}{N}, \qquad \frac{1}{(\bar{q})_i} = \frac{(q)_{N-1-i}}{N}. \tag{11}$$

3.2. Figure-eight knot 4_1. For the figure-eight knot K,

$$J_N(K) = \sum_{i=0}^{N-1}(q)_i(\bar{q})_i \tag{12}$$

Since $(q)_i(\bar{q})_i$ is a positive real number, numerical computation of the summension may have good accuracy. By using "pari-Gp 2.0.14" [5], which uses 28 digits for real numbers, $J_{N+1}/J_N(K)$ is computed by the following program.

Program. The feature of this program is to compute the formula consisting of a sum of the terms of $(q)_i$ and $(\bar{q})_i$ as a polynomial modulo $x^N - 1$. We replace q by an indeterminate x and compute everything as a polynomial in x modulo $x^N - 1$. Here we use the function of Pari to handle polynomials modulo a palynomial. At the end of the computation of $J_N(K)$, x is replaced by $q = \exp 2\,\pi\,\sqrt{-1}/N$.

The following program is for $N = 40$ case. The parameter `l` represents a list for

$$(x)_i \mod x^N - 1.$$

The i-th component of `l` is $(x)_{i-1}$. Similarly, the papareter `lm` is a list for

$$(x^{-1})_i \mod x^N - 1,$$

and `labs` for

$$(x)_{i-1}\,(x^{-1})_i \quad \mathrm{mod} \;\; x^N - 1.$$

The parameters `ansn1` and `ansn2` contain the value of $J_N(K)$ and J_{N+1} respectively. In Pari, a pollynomial $P(x)$ modulo a polynomial $Q(x)$ is represented by

```
Mod(P(x), Q(x))
```

and the part `P(x)` is obtained by

```
component(Mod(P(x), Q(x)), 2)
```

By using the above conventions, the program to compute $J_{N+1}(K)/J_N(K)$ is the following.

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(i=0, N-1, labs[i+1])
ansn1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(i=0, N-1, labs[i+1])
ansn2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ansn2/ansn1)
```

Results. The results of $\dfrac{J_{N+1}(K)}{J_N(K)}$ are obtained as in Table 1.

N	$2\,\pi\ \log\left(J_{N+1}(K)/J_N(K)\right)$
2	6.0036554722383232922231589540
3	4.5923016548771673123926186 58
4	3.9306636826923568537380649 34
5	3.5639166335413261272857696 53
6	3.3331647094247148658705627 62
7	3.1723179479388729369125278 68
8	3.0511447218477261549533747 41
9	2.9547820087697673021402997 24
10	2.8753586716024684616921969 49
11	2.8083425269428692418168376 90
12	2.7508858899022061105627494 97
13	2.7010513297304515498284359 63
15	2.6189700654185659235533265 05
17	2.5543325873663097202341508 67
20	2.4798538324247841110712562 98
30	2.3347752588851312294773776 74
40	2.2603214854773609722945688 43
50	2.2150727439035436283823130 96
60	2.1846710813938731498427125 20
70	2.1628411339802273448196500 89
80	2.1464064093466491711233251 40
90	2.1335870819311106083225663 84
100	2.1233085184934993198088210 58
110	2.1148835404569060005987328 13
120	2.1078522605788434969811923 38
130	2.1018952942195051336038072 49
140	2.0967839133635728919727800 32
150	2.0923500139532439786209028 84
160	2.0884672797437704334335903 37
180	2.0819896886343032990752406 85
200	2.0768018892797510105833290 74
220	2.0725535428463798209833707 78
250	2.0674510354441705661348313 07
280	2.0634384869601972342223534 49
310	2.0602003597420500807695315 49
350	2.0567441922448779579869994 00
380	2.0546284617805998678926550 99
410	2.0528216888717501226450272 57
440	2.0512608038480479836283098 57
470	2.0498988097465133601714390 63
500	2.0486999690186915840592219 85
600	2.0455684010209932847439791 75
700	2.0433304499426042669028722 42
1000	2.0392997935407537677539240 93
$2\pi\log\frac{J_{1001}(K)}{J_{1000}}-\frac{3\pi}{1000}$	2.0298750155799843880385361 63
$\mathrm{Vol}(K)$	2.0298832128193072500424051 09

TABLE 1

Graph. The points $\left(\frac{1}{N}, \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the above data is plotted as follows.

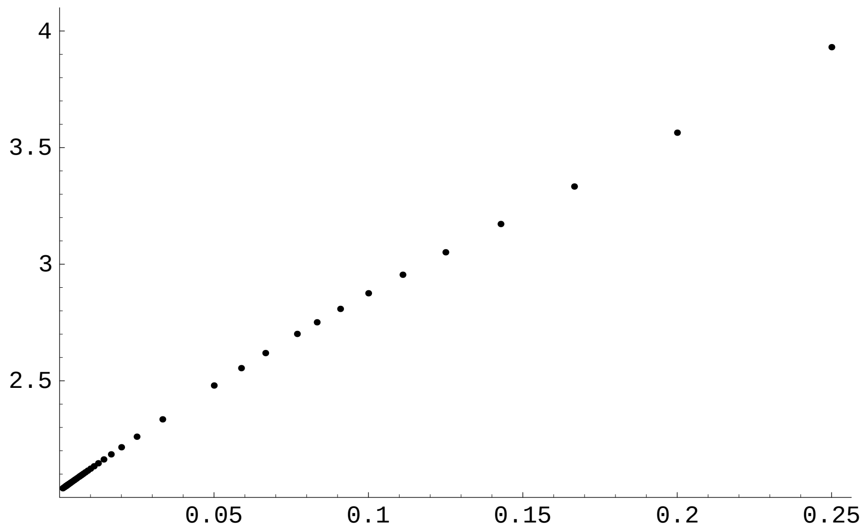


FIGURE 1. Plotting of the points $\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 4_1.

Fitting. From the above result, we can predict the actual limit by estimating the asymptotics of $J_N(K)$ by fitting with certain function, which is determined by the least square method. Here, we try to use the function of the form

$$a_0 + \frac{a_1}{N} + \frac{a_2}{N^2}. \tag{13}$$

a_0, a_1 and a_2 are obtained by the following function of Matehmatica

```
Fit[l /.  {x_, y_} -> {1/x, y}, {1, x, x^2}, x]
```

where `l` is a list of the pairs

$$\{N, J_{N+1}(K)/J_N(K)\}$$

in the above table with $N \geq 40$. The result of the fitting is

$$2.02988 + 9.42629\,\frac{1}{N} - 8.34016\,\frac{1}{N^2}.$$

Note that the constant term is equal to the volume of $S^3 \setminus K$ up to 6 digits, and the coefficient of `x` is almost equal to $3\pi = 9.42478\ldots$.

3.3. **Knot 5_2.** Let K be the knot 5_2. Then

$$J_N(K) = \sum_{i=1}^{N-1} \sum_{j=1}^{i} \frac{{(q)_i}^2}{(\bar{q})_j}.$$

This knot is achiral and $J_N(K)$ has a non-trivial imaginary part. The following results suggest that

$$\lim_{N\to\infty} \log \frac{J_{N+1}(K)}{J_N(K)} = \mathrm{Vol}(K) + \sqrt{-1}\ \mathrm{CS}(K),$$

where

$$\mathrm{Vol}(K) = 2.82812208833, \qquad \mathrm{CS}(K) = -3.02412837657.$$

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
l2 = listcreate(N)
for(i=1, N, listinsert(l2, l[i]*l[i], i))
ans = sum(i=0, N-1, l2[i+1]*sum(j=0,i, \
  l[N-1-j+1]*Mod(x^component(Mod(-j*(i+1),N), 2), x^N-1)))
ansn1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
l2 = listcreate(N)
for(i=1, N, listinsert(l2, l[i]*l[i], i))
ans = sum(i=1, N-2, l2[i+1]*sum(j=0,i, \
  l[N-1-j+1]*Mod(x^component(Mod(-j*(i+1),N), 2), x^N-1)))
ansn2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ansn2*(N-1)/ansn1/N)
```

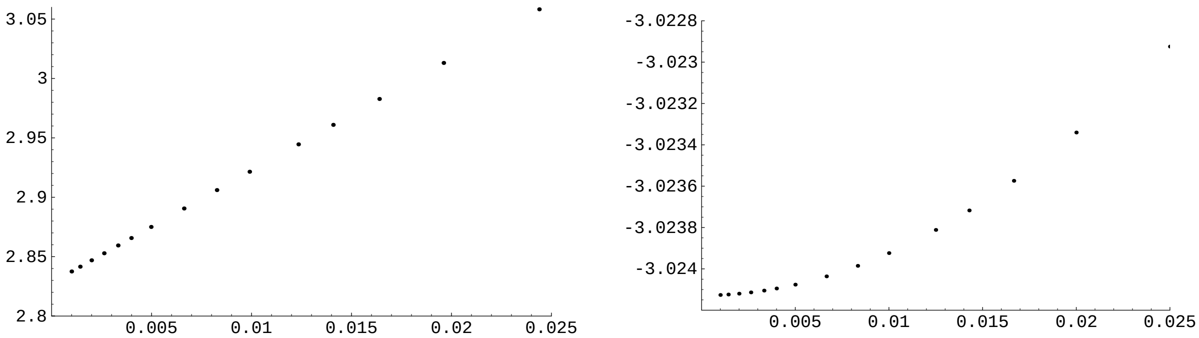
At the last line, `N` and `N-1` are added since, in the computation of `ans`, we use the relation (11).

The results are given in Table 2.

N	$2\,\pi\,\log\dfrac{J_{N+1}(K)}{J_N(K)}$
40	3.058223721261842722613885956 - 3.022924613281720287391974968 $\sqrt{-1}$
50	3.013081508530188353573854822 - 3.023340368517507069134855780 $\sqrt{-1}$
60	2.982744318753580696821772299 - 3.023574042878935429645720640 $\sqrt{-1}$
70	2.960955404961739170749114151 - 3.023717381786374852930574631 $\sqrt{-1}$
80	2.944548269170450112446966301 - 3.023811574968472287718611711 $\sqrt{-1}$
100	2.921483906108228993018469212 - 3.023923719027833555669502480 $\sqrt{-1}$
120	2.906046421388666000282542398 - 3.023985374930307234443986632 $\sqrt{-1}$
150	2.890559881907537128372001511 - 3.024036295143969179028770901 $\sqrt{-1}$
200	2.875024234226941620327156350 - 3.024076266558545340852410631 $\sqrt{-1}$
250	2.865679250969538531562099056 - 3.024094905811349375139149331 $\sqrt{-1}$
300	2.859439423619654229923900269 - 3.024105077353483138303449159 $\sqrt{-1}$
380	2.852862676601409465863918924 - 3.024113818437089706026655831 $\sqrt{-1}$
500	2.846936140234120797452382677 - 3.024119948850536105286710779 $\sqrt{-1}$
$2\pi\log\frac{J_{501}(K)}{J_{500}(K)}-\frac{3\pi}{500}$	2.828086584312582038021606817 - 3.024119948850536105286710779 $\sqrt{-1}$
$\mathrm{Vol}(K)+$ $\sqrt{-1}\ \mathrm{CS}(K)$	2.82812208833 -3.02412837657 $\sqrt{-1}$

TABLE 2. $\mathrm{CS}(K) = 2\pi^2\,\mathrm{cs}(K)$ where $\mathrm{cs}(K)$ is the Chern-Simons inariant obtained by SnapPea.

Graphs. The real and imaginary parts of the points $\left(\dfrac{1}{N}, 2\pi\ \log\dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.



real part　　　　imaginary part

FIGURE 2. Plotting of the points $\left(\dfrac{1}{N}, 2\pi\ \log\dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 5_2.

Fitting. The result of the fitting is

$$2.82812 - 3.02413\sqrt{-1} + \left(9.42398 + 0.00306762\sqrt{-1}\right)\ \frac{1}{N} - \left(8.80117 - 1.82018\sqrt{-1}\right)\ \frac{1}{N^2}.$$

3.4. **Knot** 6_1. Let K be the knot 6_1. Then

$$J_N(K) = \sum_{\substack{0\le m\le N-1 \\ 0\le k+l\le m}} \frac{|(q)_m|^2}{(\bar{q})_k\,(q)_l}\, q^{(m-k-ll)(m-k+1)}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

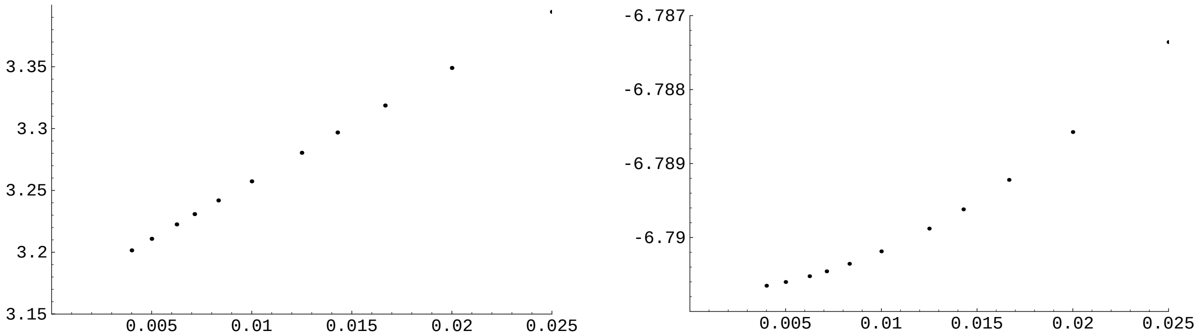
```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[m+1]*\
  sum(k=0, m, lm[N-k-1+1]*\
  sum(ll=0, m-k, l[N-ll-1+1]*\
  Mod(x^component(Mod((m-k-ll)*(m-k+1), N), 2), x^N-1)\
  )))
ans1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[m+1]*\
  sum(k=0, m, lm[N-k-1+1]*\
  sum(ll=0, m-k, l[N-ll-1+1]*\
  Mod(x^component(Mod((m-k-ll)*(m-k+1), N), 2), x^N-1)\
  )))
ans2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ans2*(N-1)^2/ans1/N^2)
```

The results are given in Table 3

N	$2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}$
40	3.3944144011894346063826743O3 - 6.78735738351777195043177974 8$\sqrt{-1}$
50	3.3490425776380577923693113 22 - 6.7885738305495517696854883 16$\sqrt{-1}$
60	3.3186638996365763346524285 37 - 6.7892205889294129903207664 23$\sqrt{-1}$
70	3.2968531397149083935140564 69 - 6.7896190525134184825149381 02$\sqrt{-1}$
80	3.2804319352095786826177059 35 - 6.7898793590085316217257410 68$\sqrt{-1}$
100	3.2573515258050954074900694 56 - 6.7901872225880429705446646 90$\sqrt{-1}$
120	3.2419056093509141651025036 53 - 6.7903554203518230548667037 34$\sqrt{-1}$
140	3.2308438371183693722904522 65 - 6.7904572370722887340263144 03$\sqrt{-1}$
160	3.2225316285108986636900218 23 - 6.7905235088670618998477316 03$\sqrt{-1}$
200	3.2108716264003887442166039 73 - 6.7906016694041463045053554 65$\sqrt{-1}$
250	3.2015243454482753800031168 10 - 6.7906518461046179070622170 87$\sqrt{-1}$
$2\pi \log \frac{J_{251}(K)}{J_{250}(K)} - \frac{3\pi}{250}$	3.1638252336051978611415650 89 - 6.7906518461046179070622170 87$\sqrt{-1}$
$\mathrm{Vol}(K)+$ $\sqrt{-1}\,\mathrm{CS}(K)$	3.1639632289 -6.7907414993 $\sqrt{-1}$

TABLE 3. $\mathrm{CS}(K) = -2\pi^2 \operatorname{cs}(K) + \pi^2$.

Graphs. The real and imaginary parts of the points $\left(\dfrac{1}{N}, \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.



real part imaginary part

FIGURE 3. Plotting of the points $\left(\dfrac{1}{N}, \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 6_1.

Fitting. The result of the fitting is

$$\left(3.16404 - 6.79075\sqrt{-1}\right) + \left(9.40652 + 0.00370946\sqrt{-1}\right)\,\frac{1}{N} - \left(7.70212 - 5.27915\sqrt{-1}\right)\,\frac{1}{N^2}$$

3.5. **Knot 6_3.** Let K be the knot 6_3. Then

$$J_N(K) = \sum_{\substack{k,l,m \geq 0 \\ k+l+m \leq N-1}} \left| \frac{(q)_{k+l+m}}{(\bar{q})_l \, (q)_m} \right|^2 (q)_{k+l} \, (\bar{q})_{m+k} \, q^{(m-l)(k+1)}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[N-1-m+1]*\
  sum(p=0, N-1-m, labs[p+m+1]*lm[p+1]*\
  sum(k=0, p, labs[N-1-p+k+1]*l[m+k+1]*\
  Mod(x^component(Mod(-(m-p+k)*(k+1), N), 2), x^N-1))))
ans1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[N-1-m+1]*\
  sum(p=0, N-1-m, labs[p+m+1]*lm[p+1]*\
  sum(k=0, p, labs[N-1-p+k+1]*l[m+k+1]*\
  Mod(x^component(Mod(-(m-p+k)*(k+1), N), 2), x^N-1))))
ans2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ans2*(N-1)^4/ans1/N^4)
```

The results are given in Table 4.

N	$2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}$
30	5.991757632388930862295686837
40	5.920010510909063767712690688
47	5.887310870362322038241138727
50	5.876009180047075973936088402
60	5.846282921844738453303387249
70	5.824859282414985211663083205
80	5.808687819822659085249294793
94	5.791733883431946311125566885
100	5.785898993155213353224223121
120	5.770610335748061213979602476
150	5.755245033266310556638346366
$2\pi \log \frac{J_{251}(K)}{J_{250}(K)} - \frac{3\pi}{150}$	5.692413180194514691869093498
$\mathrm{Vol}(K)$	5.69302109128

TABLE 4

Graph. The real and imaginary parts of the points $\left(\dfrac{1}{N}, 2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.

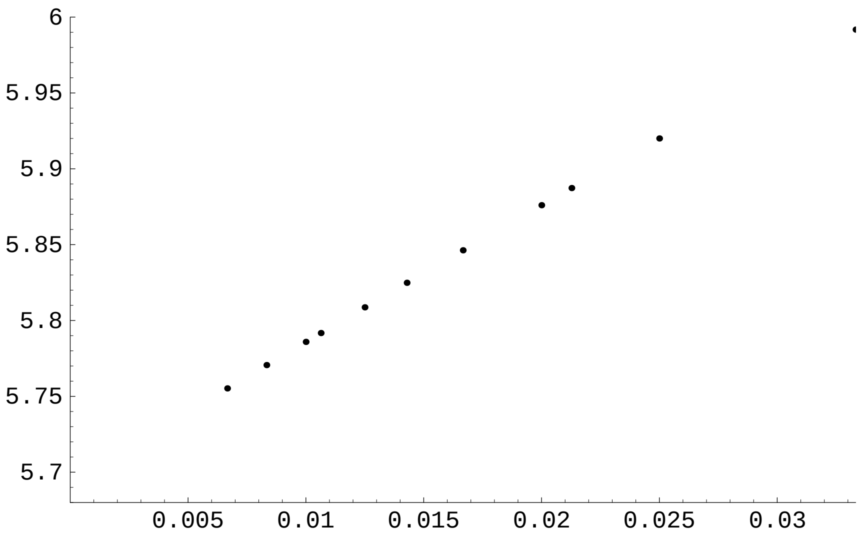


FIGURE 4. Plotting of the points $\left(\dfrac{1}{N}, 2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 6_3.

Fitting. The result of the fitting is

$$5.69297 + 9.43385\,\frac{1}{N} - 14.1061\,\frac{1}{N^2}.$$

3.6. **Knot** 8_9**.** Let K be the knot 8_9. Then

$$J_N(K) = \sum_{\substack{0 \le l, m_1, m_2, n_1, n_2 \le N-1 \\ m_1+n_1, m_2+n_2 \le l \\ m_1+m_2 \le l}} \left| \frac{(q)_{l-m_1}(q)_l(q)_{l-m_2}}{(q)_{m_1}(q)_{m_2}(q)_{n_1}(q)_{n_2}} \right|^2 \frac{(\bar{q})_{l-n_1}(q)_{l-n_2}}{(q)_{l-m_1-n_1}(\bar{q})_{l-m_2-n_2}} \times q^{(m_2-m_1)(l-m_1-m_2)+(n_2-n_1)(l-n_1-n_2)+m_2-m_1+n_2-n_1}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ is almost equal to those for the previous examples. The only different lines are the following.

```
...

ans = sum(ll=0, N-1, labs[ll+1]*\
 sum(m1=0, ll, labs[ll-m1+1]*labs[N-1-m1+1]*\
  sum(n1=0, ll-m1, labs[N-1-n1+1]*lm[ll-n1+1]*lm[N-1-ll+m1+n1+1]*\
   sum(m2=0, ll-m1, labs[ll-m2+1]*labs[N-1-m2+1]*\
    sum(n2=0, ll-m2, labs[N-1-n2+1]*l[ll-n2+1]*l[N-1-ll+m2+n2+1}*\
      Mod(x^(((m2-m1)*(ll-m1-m2)+(n2-n1)*(ll-n1-n2)+m2-m1+n2-n1)%N), \
x^N-1))))))

...

2*Pi*log(ans2*(N-1)^10/ans1/N^10)
```

The results are given in Table 5.

N	$2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}$
5	8.036805097240829695180371009
10	8.373856508425248006124939747
15	8.152791235806956158626064554
20	8.021952312877980724820244796
25	7.941218675423634478989298960
30	7.885684247868739884080382928
40	7.814415752862457696272810490
50	7.770664225432679874868903250
$2\pi \log \frac{J_{51}(K)}{J_{50}(N)} - \frac{3\pi}{50}$	7.582168666217292280561144647
$\mathrm{Vol}(K)$	7.5881802236416

TABLE 5

Graph. The points $\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.

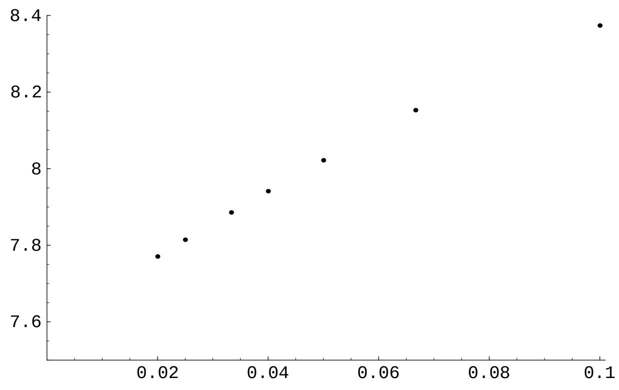


FIGURE 5. Plotting of the points $\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 8_9.

3.7. **Knot** 8_{20}**.** Let K be the knot 8_{20}. Then

$$J_N(K) = \sum_{\substack{j,l\le k\le i+l\le j+m \\ j\le i \\ 0\le i,j,k,l,m\le N-1}} \frac{\left\{(\bar{q})_i(q)_k(\bar{q})_m\right\}^2}{\left\{(\bar{q})_j(q)_l\right\}(q)_{k-l}(\bar{q})_{i-k+l}(\bar{q})_{j+m-i-l}(q)_{i-j}(q)_{k-j}}\, q^{k+m+im+km-il}.$$

Program.

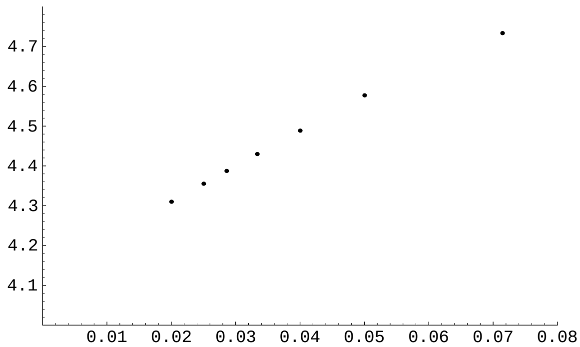
```
...

ans = sum(i=0, N-1, lm2[i+1]*\
 sum(j=0, i, l2[N-1-j+1]*lm[N-1-i+j+1]*\
  sum(ll=0, N-1, lm2[N-1-ll+1]*\
   sum(k=max(ll,j), min(N-1, i+ll), \
      l2[k+1]*lm[N-1-k+ll+1]*l[N-1-i+k-ll+1]*lm[N-1-k+j+1]*\
    sum(m=i-j+ll, N-1, lm2[m+1]*l[N-1-j-m+i+ll+1]*\
       Mod(x^((k+m+i*m+k*m-i*ll)%N), x^N-1)\
)))))

...

2*Pi*log(ans2*(N-1)^9/ans1/N^9)
```

Results.

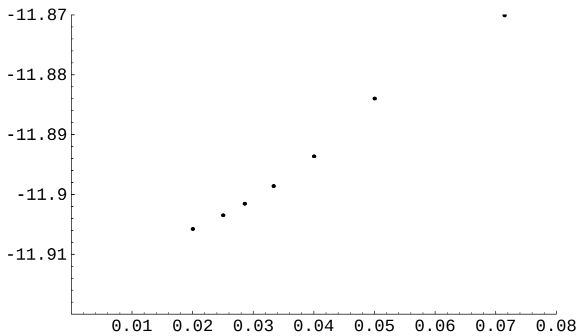
N	$2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}$
5	4.993134830282317119922109736 - 8.534810138421228059039058370$\sqrt{-1}$
7	6.058772085097703463174557594 - 13.01002462670787288866716257$\sqrt{-1}$
10	4.838146313755788700051905369 - 11.94729400926942721637213050$\sqrt{-1}$
14	4.733597316958210595817845225 - 11.87005752620625791877540683$\sqrt{-1}$
20	4.577298093617009639760204539 - 11.88396870671513344794156134$\sqrt{-1}$
25	4.488646016707939440733681448 - 11.89362552924285078082734017$\sqrt{-1}$
30	4.429868129481447562532108244 - 11.89859594888974090236089072$\sqrt{-1}$
35	4.387402367809206482628251653 - 11.90153927773429954376145556$\sqrt{-1}$
40	4.355311811237171523425351164 - 11.90347183192930081507773909$\sqrt{-1}$
50	4.310046749251591944060510784 - 11.90576469622971586668761426$\sqrt{-1}$
$2\pi\log\frac{J_{51}(K)}{J_{50}(N)} - \frac{3\pi}{50}$	4.121551190036204349752752181 - 11.90576469622971586668761426$\sqrt{-1}$
Vol(K)+ $\sqrt{-1}$ CS(K)	4.1249032518077 -11.9099170709 $\sqrt{-1}$

Table 6. $\mathrm{CS}(K) = -2\pi^2\,\mathrm{cs}(K) - \pi^2$.

Graphs.



real part



imaginary part

Figure 6. Plotting of the points $\left(\dfrac{1}{N}, 2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 8_{20}.

3.8. **Whitehead link.** The Whitehead link is the most simplest hyperbolic 2-component link. It is not a one-component knot, but complexified hyperbolic volume conjecture seems to hold for this link as follows.

Let K be the Whitehead link. Then

$$J_N(K) = \sum_{\substack{0 \le i,j,k \le N-1 \\ k \le i,j}} \frac{\left\{(\bar{q})_i (\bar{q})_j\right\}^2}{{(q)_k}^4 (\bar{q})_{i-k} (\bar{q})_{j-k}}\, q^{-(N-1)N/2}$$

Program.

```
...

ans = sum(k=0, N-1, lm4[N-1-k+1]*\
        sum(i=k, N-1, lm2[i+1]*l[N-1-i+k+1]*\
          sum(j=k, N-1, lm2[j+1]*l[N-1-j+k+1])))

...

2*Pi*log(ans2*(N-1)^6/ans1/N^6)
```

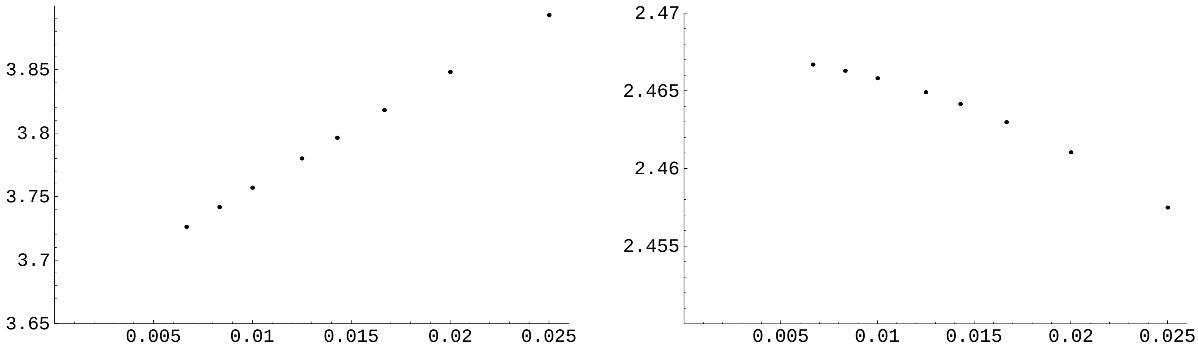
Here $q^{-(N-1)N/2}$ is omitted since it is equal to ± 1, which contributes to $\mathrm{CS}(K)$ by a multiple of $2\pi^2$.

Results.

N	$2\pi \ \log \dfrac{J_{N+1}(K)}{J_N(K)}$
40	$3.8929203591018110978095255583 + 2.4574839973308660458125047 03\sqrt{-1}$
50	$3.8481614664029142251545301 80 + 2.4610394740180165698697453 01\sqrt{-1}$
60	$3.8180290133494993127082361 53 + 2.4629767486759802547033908 55\sqrt{-1}$
70	$3.7963625012095376910789445 56 + 2.4641471917958816145824764 51\sqrt{-1}$
80	$3.7800343275600221950820153 85 + 2.4649079234047646222743958 68\sqrt{-1}$
100	$3.7570622589854778572479912 39 + 2.4658037859628196792363273 39\sqrt{-1}$
120	$3.7416746081790236731591442 58 + 2.4662910858966602606886061 42\sqrt{-1}$
150	$3.7262286497265585905078284 29 + 2.4666902040110300079621138 80\sqrt{-1}$
$2\pi \log \frac{J_{151}(K)}{J_{150}(N)} - \frac{3\pi}{150}$	$3.6633967966547627257385755 61 + 2.4666902040110300079621138 80\sqrt{-1}$
$\mathrm{Vol}(K) + \sqrt{-1}\ \mathrm{CS}(K)$	$3.6638623767089 + 2.46740110027234\ \sqrt{-1}$

TABLE 7. $\mathrm{CS}(K) = 2\pi^2\, \mathrm{cs}(K)$, where $cs(K) = 1/8$.

Graphs.



real part imaginary part

FIGURE 7. Plotting of the points $\left(\dfrac{1}{N}, 2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the Whitehead link.

Fitting.

$$3.66386 + 2.46742\sqrt{-1} + \left(9.42575 - 0.00497353\sqrt{-1}\right)\frac{1}{N} - \left(10.5298 + 15.7048\sqrt{-1}\right)\frac{1}{N^2}$$

Remark. The volume conjecture (4) is not hold for all links, because $J_N(L) = 0$ if L is a split link $L = K_1 \sqcup K_2$. In this case,

$$\left|S^3 \setminus L\right| = \left|S^3 \setminus K_1\right| + \left|S^3 \setminus K_2\right|$$

and so

$$\lim_{N \to \infty} \exp\left(N \left|S^3 \setminus L\right|\right)$$

does not equal to 0 if K_1 and K_2 are both hyperbolic knots.

4. Conclusion

In the above computations, we see the behavior of $\dfrac{J_{N+1}(K)}{J_N(K)}$ to chech the formula (6). We also compare with Hikami's obserbation (7). Both conjectures (6) and (7) seem to be true for the examples given here. Moreover, the imaginary part of the coefficient of $\dfrac{1}{N}$ in the asymptotic expansion of $\dfrac{J_{N+1}(K)}{J_N(K)}$ seems to be 0 for these examples.

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