

4 Proof of invariance

Thm.3 $\mathcal{H}^{i,j}(L)$ is a link invariant. ($i, j \in \mathbb{Z}$)

Lemma 2. C : chain complex C' : subcomplex of C

$$(1) \ C' : \text{acyclic} \implies H^*(C) \stackrel{j^*}{\cong} H^*(C/C').$$

$$(2) \ C/C' : \text{acyclic} \implies H^*(C) \stackrel{i^*}{\cong} H^*(C').$$

Lemma 3. (1) $\mathcal{H}_D^*(C) := \text{Ker } \partial / \text{Im } \partial$.

$$\implies \mathcal{H}_D^*(C) \cong \bigoplus_{i,j \in \mathbb{Z}} \mathcal{H}_D^{i,j}(C).$$

$$(2) \ f : C(D) \longrightarrow C'(D) \quad \text{chain map,}$$

$$f^* : \mathcal{H}_D^*(C) \longrightarrow \mathcal{H}_D^*(C') \quad \text{isomorphism.}$$

$$\implies \mathcal{H}_D^{i,j}(C) \cong \mathcal{H}_D^{i,j}(C') (i, j \in \mathbb{Z}).$$

The outline of this proof

1. Invariance under changing an order given to the crossing points of D

$(C(D), \partial)$: a chain complex

$(C'(D), \partial')$: The chain complex given by exchanging the i th crossing point for $(i + 1)$ th one of D

We'll change the signs of some generators of C .

The incidence number of ∂ will equal to the incidence number of ∂' at any (S_l, S_m) .

2. Invariance under the Reidemeister moves

We'll reduce some generators of $C(D)$ by using Lemma 2, 3.

Independence of changing the order

S^{*11*} : An enhanced state s.t. 1-smoothing is given to the i th, $(i + 1)$ th crossing points of D

We'll take $-S^{*11*}$ as a generator of $C(D)$ for S^{*11*} .

(i) $k = i$, $S_1 \in *01*$, $S_2 \in *11*$.

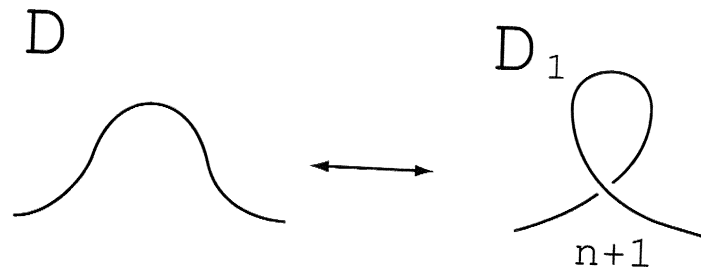
(ii) $k = i + 1$, $S_1 \in *10*$, $S_2 \in *11*$.

(iii) $k \neq i, i + 1$, $S_1, S_2 \in *11*$.

(iv) $k \neq i, i + 1$, $S_1, S_2 \in *00*$, $*01*$, $*10*$, or

$S_1 \in *00*$, $S_2 \in *01*$ or $S_2 \in *10*$.

Invariance under R1

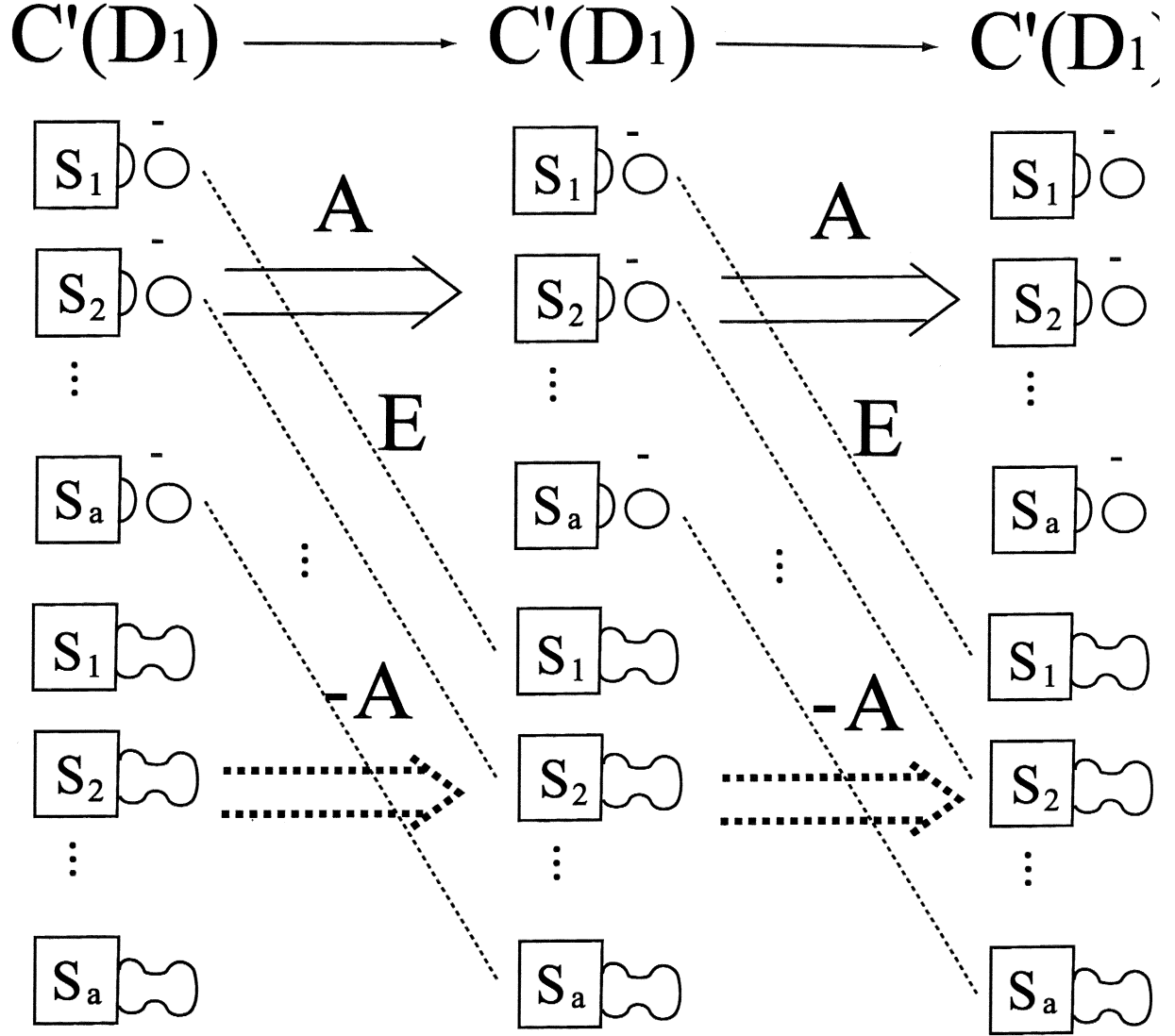


$$C(D_1) = \langle \boxed{S_p} \circlearrowleft^+, \boxed{S_q} \circlearrowleft^-, \boxed{S_r} \text{ (figure-eight)} \rangle .$$

$$C'(D_1) := \langle \boxed{S_q} \circlearrowleft^-, \boxed{S_r} \text{ (figure-eight)} \rangle .$$

We'll prove that $C'(D_1)$ is an acyclic subcomplex.

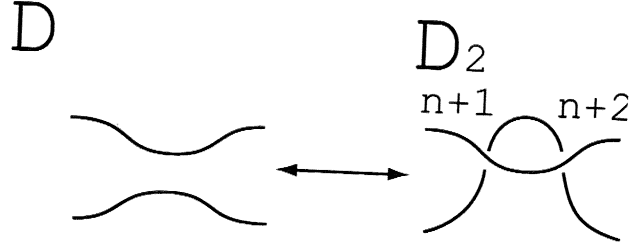
$$(\because i(\boxed{S} \circlearrowleft^+) = i(\boxed{S} \circlearrowleft), j(\boxed{S} \circlearrowleft^+) = j(\boxed{S} \circlearrowleft))$$



The presentation matrix of $\partial|_{C'(D_1)}$ is $\begin{pmatrix} A & O \\ E & -A \end{pmatrix}$,

$\text{Ker} \partial|_{C'(D_1)} = \text{Im} \partial|_{C'(D_1)}$, i.e. $C'(D_1)$ is acyclic. $\square$

Proof of invariance under R2



$$C(D_2) = \langle \boxed{\text{diagram}}^{S_1^{p00}}, \boxed{\text{diagram}}^{S_2^{q01}}, \boxed{\text{diagram}}^{S_3^{r10}}, \boxed{\text{diagram}}^{S_4^{t11}} \rangle.$$

$$C'(D_2) := \langle \boxed{\text{diagram}}^{S_2^{q01}}, \boxed{\text{diagram}}^{S_4^{t11}} \rangle$$

$C'(D_2)$ is an acyclic subcomplex of $C(D_2)$.

$$\therefore \mathcal{H}_D^*(C) \cong \mathcal{H}_D^*(C/C').$$

$$C/C' \cong \langle \boxed{\text{diagram}}^{S_1^{p00}}, \boxed{\text{diagram}}^{S_2^{q01}}, \boxed{\text{diagram}}^{S_3^{r10}} \rangle. \quad C'' := \langle \boxed{\text{diagram}}^{S_3^{r10}} \rangle,$$

$$C/C'/C'' \cong \langle \boxed{\text{diagram}}^{S_1^{p00}}, \boxed{\text{diagram}}^{S_2^{q01}} \rangle.$$

$C/C'/C''$ is an acyclic subcomplex of C/C' .

$$\therefore \mathcal{H}_D^*(C/C') \cong \mathcal{H}_D^*(C'').$$

$$\therefore \mathcal{H}_D^*(C) \cong \mathcal{H}_D^*(C'').$$

$$(i(\boxed{\text{S}}) = i(\boxed{\text{S}}), j(\boxed{\text{S}}) = j(\boxed{\text{S}})) \quad \square$$

Second proof of invariance under R2

We'll start it from C/C' .

$$\partial_{00}^{01}(\boxed{\text{S}_1^{*00}}) := \boxed{\text{S}_1^{*01}}, \quad \partial_{00}^{10}(\boxed{\text{S}_1^{*00}}) := \boxed{\text{S}_2^{*10}}$$

$$C''' := \langle \boxed{\text{S}_1^{\text{p}00}}, \boxed{\text{S}_1^{\text{p}01}} + \partial_{00}^{01} \circ (\partial_{00}^{10})^{-1}(\boxed{\text{S}_1^{\text{p}01}}) \rangle.$$

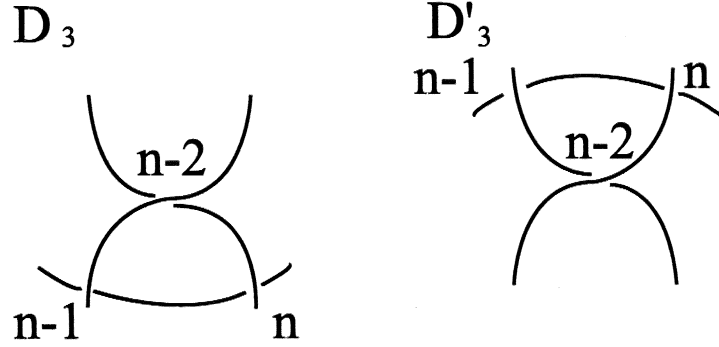
C''' is an acyclic subcomplex of C/C' .

$$\therefore \mathcal{H}_D^*(C/C') \cong \mathcal{H}_D^*(C/C'/C''').$$

$$\therefore \mathcal{H}_D^*(C) \cong \mathcal{H}_D^*(C/C'/C''').$$

$$(C/C'/C''' \cong \langle \boxed{\text{S}_3^{*10}} \rangle)$$

Invariance under R3



$$C(D_3)$$

$$= \langle S_1^{p*000}, S_2^{q*001}, S_3^{r*010}, S_4^{t*011}, S_5^{u*100}, S_6^{v*101}, S_7^{w*110}, S_8^{x*111} \rangle,$$

$$C(D'_3)$$

$$= \langle S_1^{p*000}, S_2^{q*001}, S_3^{r*010}, S_4^{t*011}, S_5'^{u*100}, S_7^{w*101}, S_6'^{v*110}, S_8'^{x*111} \rangle.$$

$$C'(D_3) := \langle S_5^{u*100}, S_6^{v*101}, S_7^{w*110}, S_8^{x*111} \rangle,$$

$$C'(D'_3) := \langle S_5'^{u*100}, S_7^{w*101}, S_6'^{v*110}, S_8'^{x*111} \rangle.$$

We'll reduce these subcomplices in the same way as the second proof of $R2$.

$$C(D_3)$$

$$= \langle \begin{matrix} \mathbf{a}_1^p & \mathbf{a}_2^q & \mathbf{a}_3^r & \mathbf{a}_4^t & \mathbf{a}_5^u & \mathbf{a}'^u_5 \\ \mathbf{s}_1^p & \mathbf{s}_2^q & \mathbf{s}_3^r & \mathbf{s}_4^t & \mathbf{s}_5^u & \mathbf{s}_6^u \end{matrix} \begin{matrix} *000 & *001 & *010 & *011 & *110 & *101 \end{matrix} \begin{matrix} \boxed{\text{diagram 1}} & \boxed{\text{diagram 2}} & \boxed{\text{diagram 3}} & \boxed{\text{diagram 4}} & \boxed{\text{diagram 5}} & \boxed{\text{diagram 6}} \end{matrix} \rangle,$$

$$C(D'_3)$$

$$= \langle \begin{matrix} \mathbf{b}_1^p & \mathbf{b}_2^q & \mathbf{b}_3^r & \mathbf{b}_4^t & \mathbf{b}_5^u & \mathbf{b}'^u_5 \\ \mathbf{s}_1^p & \mathbf{s}_2^q & \mathbf{s}_3^r & \mathbf{s}_4^t & \mathbf{s}_5^u & \mathbf{s}_6^u \end{matrix} \begin{matrix} *000 & *001 & *010 & *011 & *101 & *110 \end{matrix} \begin{matrix} \boxed{\text{diagram 1}} & \boxed{\text{diagram 2}} & \boxed{\text{diagram 3}} & \boxed{\text{diagram 4}} & \boxed{\text{diagram 5}} & \boxed{\text{diagram 6}} \end{matrix} \rangle.$$

$$f : C(D_3) \longrightarrow C(D'_3) \quad \text{homomorphism}$$

$$f(a_i^k) := \begin{cases} b_i^k & (i=1, 2, 3, 4) \\ -b_5^k & (i=5) \end{cases}$$

(If we get a_5^k, b_5^k , we'll change them for $-a_5^k, -b_5^k$ respectively.)

We'll obtain that f is an isomorphic chain map. $\square$