4 Proof of invariance

Thm.3 $\mathcal{H}^{i,j}(L)$ is a link invariant.$(i, j \in \mathbb{Z})$

Lemma 2. C : chain complex C' : subcomplex of C

(1) C' : acyclic $\Longrightarrow H^*(C) \overset{j^*}{\cong} H^*(C/C')$.

(2) C/C' : acyclic $\Longrightarrow H^*(C) \overset{i^*}{\cong} H^*(C')$.

Lemma 3.(1) $\mathcal{H}^*_D(C) := Ker\partial / Im\partial$.

$\Longrightarrow \mathcal{H}^*_D(C) \cong \bigoplus_{i,j\in\mathbb{Z}} \mathcal{H}^{i,j}_D(C)$.

(2) $f : C(D) \longrightarrow C'(D)$ chain map,

$f^* : \mathcal{H}^*_D(C) \longrightarrow \mathcal{H}^*_D(C')$ isomorphism.

$\Longrightarrow \mathcal{H}^{i,j}_D(C) \cong \mathcal{H}^{i,j}_D(C')(i, j \in \mathbb{Z})$.

The outline of this proof

1. Invariance under changing an order given to the crossing points of D

$(C(D), \partial)$: a chain complex

$(C'(D), \partial')$: The chain complex given by exchanging the ith crossing point for $(i+1)$th one of D

We'll change the signs of some generators of C. The incidence number of ∂ will equal to the incidence number of ∂' at any (S_l, S_m).

2. Invariance under the Reidemeister moves

We'll reduce some generators of $C(D)$ by using Lemma 2, 3.

Independence of changing the order

S^{*11*} : An enhanced state s.t. 1-smoothing is given to the ith, $(i+1)$th crossing points of D

We'll take $-S^{*11*}$ as a generator of $C(D)$ for S^{*11*}.

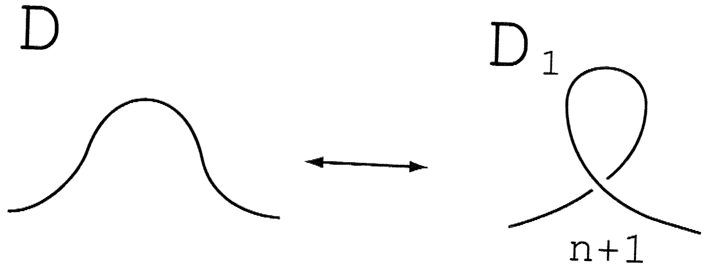
(i)$k = i$, $S_1 \in *01*, S_2 \in *11*$.

(ii)$k = i+1$, $S_1 \in *10*, S_2 \in *11*$.

(iii)$k \neq i, i+1$, $S_1, S_2 \in *11*$.

(iv)$k \neq i, i+1$, $S_1, S_2 \in *00*, *01*, *10*$, or $S_1 \in *00*, S_2 \in *01*$ or $S_2 \in *10*$.

Invariance under R1

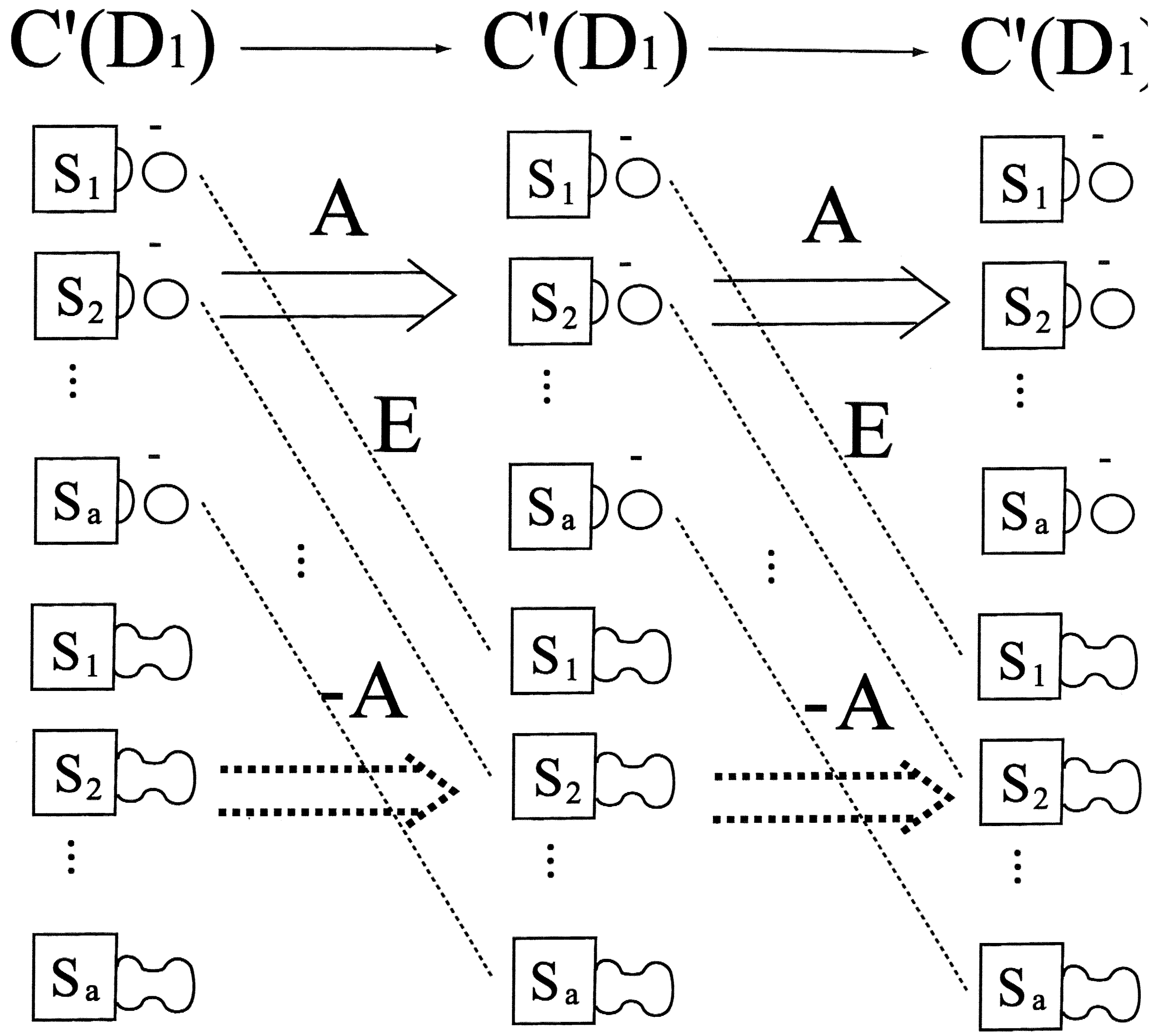


$$C(D_1) =\langle \boxed{S_p}\,\overset{+}{\bigcirc},\ \boxed{S_q}\,\overset{-}{\bigcirc},\ \boxed{S_r}\bigcirc\!\!\bigcirc \rangle .$$

$$C'(D_1) :=\langle \boxed{S_q}\,\overset{-}{\bigcirc},\ \boxed{S_r}\bigcirc\!\!\bigcirc \rangle .$$

We'll prove that $C'(D_1)$ is an acyclic subcomplex.

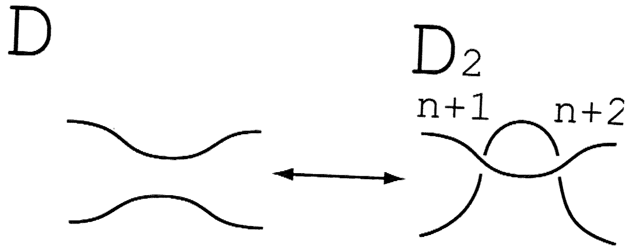
$$(\because\ i(\boxed{S}\,\overset{+}{\bigcirc}) = i(\boxed{S}), j(\boxed{S}\,\overset{+}{\bigcirc}) = j(\boxed{S})\ )$$

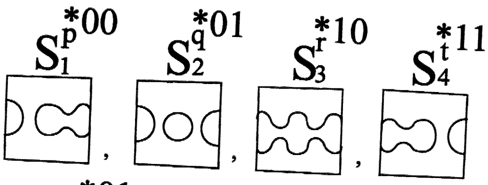


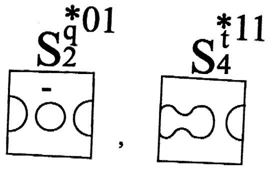
The presentation matrix of $\partial|_{C'(D_1)}$ is $\begin{pmatrix} A & O \\ E & -A \end{pmatrix}$,

$Ker\partial|_{C'(D_1)} = Im\partial|_{C'(D_1)}$, i.e. $C'(D_1)$ is acyclic. $\square$

Proof of invariance under R2

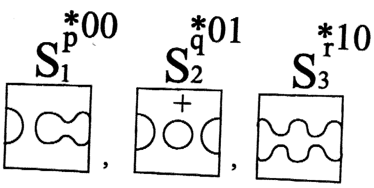
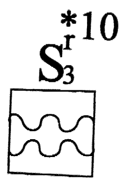


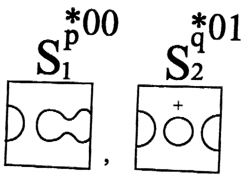
$C(D_2) =<$  $> .$

$C'(D_2) :=<$  $>$

$C'(D_2)$ is an acyclic subcomplex of $C(D_2)$.

$\therefore \mathcal{H}_D^*(C) \cong \mathcal{H}_D^*(C/C')$.

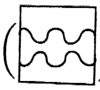
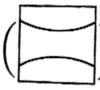
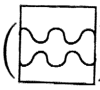
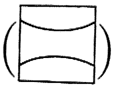
$C/C' \cong<$  $>$. $C'' :=<$  $>$,

$C/C'/C'' \cong<$  $> .$

$C/C'/C''$ is an acyclic subcomplex of C/C'.

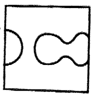
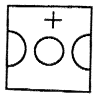
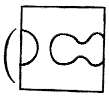
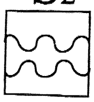
$\therefore \mathcal{H}^*_D(C/C') \cong \mathcal{H}^*_D(C'')$.

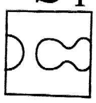
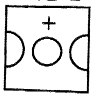
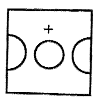
$\therefore \mathcal{H}^*_D(C) \cong \mathcal{H}^*_D(C'')$.

$(i($ S  $) = i($ S  $), j($ S  $) = j($ S  $))$ □

Second proof of invariance under R2

We'll start it from C/C'.

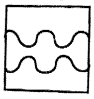
$\partial^{01}_{00}($ S_1^{*00}  $) :=$ S_1^{*01} , $\partial^{10}_{00}($ S_1^{*00}  $) :=$ S_2^{*10} 

$C''' :=<$ $S_1^{*00}_{p}$ , $S_1^{*01}_{p}$  $+ \partial^{01}_{00} \circ (\partial^{10}_{00})^{-1}($ $S_1^{*01}_{p}$  $) >$.

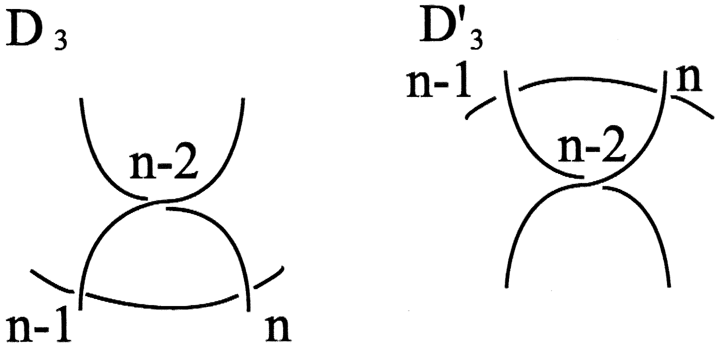
C''' is an acyclic subcomplex of C/C'.

$\therefore \mathcal{H}^*_D(C/C') \cong \mathcal{H}^*_D(C/C'/C''')$.

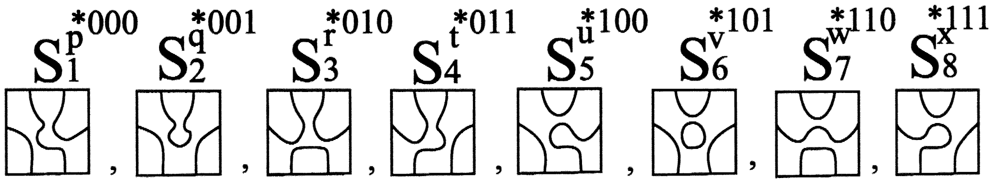
$\therefore \mathcal{H}^*_D(C) \cong \mathcal{H}^*_D(C/C'/C''')$.

$(C/C'/C''' \cong<$ $S_3^{*10}_{r}$  $>)$

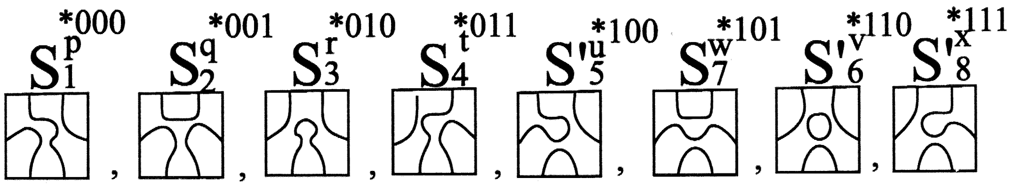
Invariance under R3

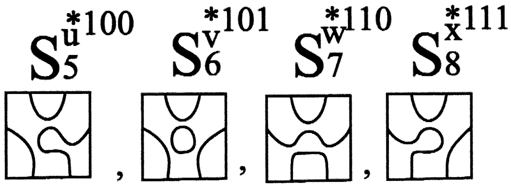


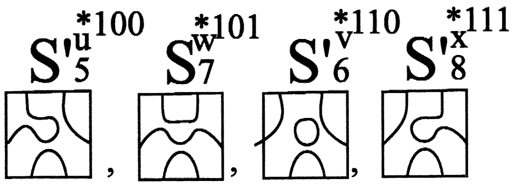
$C(D_3)$

=<  >,

$C(D_3')$

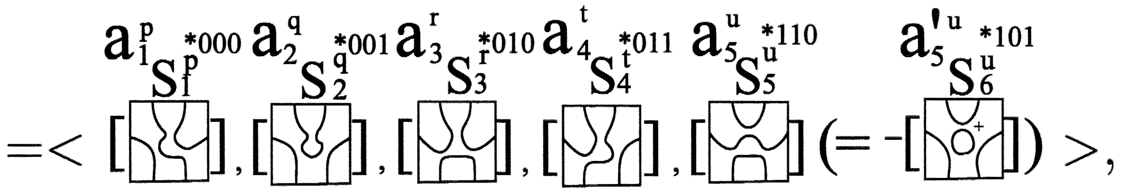
=<  >.

$C'(D_3)$:=<  >,

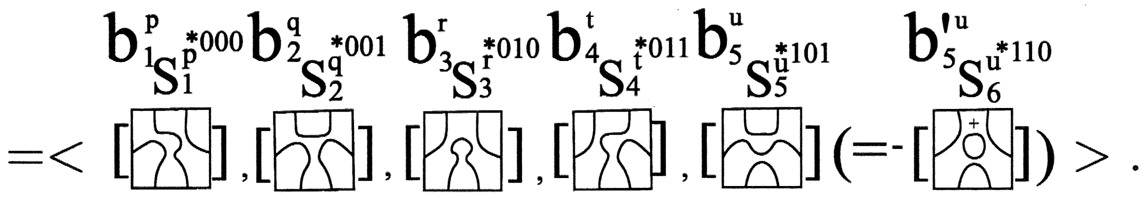
$C'(D_3')$:=<  >.

We'll reduce these subcomplices in the same way as the second proof of $R2$.

$C(D_3)$



$C(D'_3)$



$f : C(D_3) \longrightarrow C(D'_3)$ homomorphism

$$f(a_i^k) := \begin{cases} b_i^k & (i{=}1,\ 2,\ 3,\ 4) \\ -b_5^k & (i{=}5) \end{cases}$$

(If we get ${a'}_5^k, {b'}_5^k$, we'll change them for $-a_5^k$, $-b_5^k$ respectively.)

We'll obtain that f is an isomorphic chain map. $\square$